Chapter 2

Price Pseudo-Variance, Pseudo-Covariance, Pseudo-Volatility, and Pseudo-Correlation Swaps — In Analytical Closed-Forms

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2.1 Introduction

This problem was brought to us by Ritchie (Yeqi) He of RBC Financial Group. It concerns the pricing of swaps involving the so-called pseudo-statistics, namely the pseudo-variance, -covariance, -volatility, and -correlation. These products provide an easy way for investors to gain exposure to the future level of volatility. Swaps, in this context, are forward contracts in which the underlying is an interest rate S_t. They are also path-dependent because the payoffs depend on the trajectory of S_t on some time period $[T_s, T_e] \subseteq [0, T]$, where T is the maturity date. The prefix *pseudo* is used to indicate that samples of the underlying are taken at discrete times $t_0, t_1, ..., t_n \in [T_s, T_e]$. In the usual complete market framework, the unique prices of these swaps can be computed as the mathematical expectation of the discounted payoffs under the measure in which the discounted rate process is a martingale. In this report, we present analytic

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formulas for these expectations for the pseudo-variance and pseudo-volatility swaps, as requested by the problem statement. Also, we use Monte-Carlo simulation to experiment with a stochastic volatility model.

2.2 Definitions

Here we briefly describe the problem presented to us. The bibliography contains references on variance and volatility swaps. The market we consider consists of the strictly positive underlying rates $S_t^{(1)}$ and $S_t^{(2)}$ satisfying the stochastic differential equation (SDE)

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \mu_t^{(i)}\, dt + \sigma_t^{(i)}\, dW_t^i, \qquad t > 0, \qquad i = 1,\, 2$$

and a numeraire N_t which is a zero-coupon bond $\mathrm{df}(t,T)$. Here $dW_t^{(i)}$ are standard Wiener processes (zero mean, unit variance per unit time) with correlation $\rho_t dt$. In the cases of the variance and volatility swaps, we drop the superscripts so for example we write $S_t = S_t^{(1)}$ and $\mu_t = \mu_t^{(1)}$, etc. We assume that the market is complete, so there exists a unique martingale measure Q with respect to N_t. First, let us define the following continuously realized (measured) statistics over an observation period $[T_s, T_e]$:

$$\Sigma^2_{(S)}(T_s,T_e) = \frac{1}{T_e - T_s}\int_{T_s}^{T_e} \sigma_\tau^2 d\tau, \qquad \text{(realized volatility-square)}$$

$$\Sigma^2_{(S^{(1)},S^{(2)})}(T_s,T_e) = \frac{1}{T_e - T_s}\int_{T_s}^{T_e} \sigma_\tau^{(1)}\sigma_\tau^{(2)}\rho_\tau d\tau, \qquad \text{(realized volatility-cross)}$$

$$\sigma_{(S)}(T_s,T_e) = \sqrt{\frac{1}{T_e - T_s}\int_{T_s}^{T_e} \sigma_\tau^2 d\tau}, \qquad \text{(realized volatility)}$$

$$\rho_{(S^{(1)},S^{(2)})}(T_s,T_e) = \frac{\int_{T_s}^{T_e} \sigma_\tau^{(1)}\sigma_\tau^{(2)}\rho_\tau d\tau}{\sqrt{\int_{T_s}^{T_e} {\sigma_\tau^{(1)}}^2 d\tau}\sqrt{\int_{T_s}^{T_e} {\sigma_\tau^{(2)}}^2 d\tau}}. \qquad \text{(realized correlation)}$$

Note that when $S_t^{(1)} \equiv S_t^{(2)}$, the correlation is equal to one and the volatility-cross coincides with the volatility-square. Here we only consider the simpler case where we approximate the above quantities by ones that are discretely sampled. Thus, let $T_s = t_0 < t_1 < ... < t_n = T_e$ be the sampling dates, and we define the log-return[1] for the underlying rate S as

$$X_i^{(k)} = \log\left(\frac{S_{t_i}^{(k)}}{S_{t_{i-1}}^{(k)}}\right), \qquad k = 1, 2, \qquad i = 1, 2, ..., n$$

and also denote the arithmetic mean by

$$\bar{X}_n^{(k)} = \frac{1}{n}\sum_{i=1}^{n} X_i^{(k)}, \qquad k = 1, 2.$$

[1]Here log denotes the natural logarithm.

Now, we can define the following realized pseudo-statistics:

$$\hat{\Sigma}^2_{(S)}(n;T_S,T_e) = \frac{n}{T_e - T_s}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right), \qquad \text{(realized pseudo-volatility-square)}$$

$$\hat{\Sigma}^2_{(S^{(1)},S^{(2)})}(n;T_s,T_e) = \frac{n}{T_e - T_s}\left(\frac{1}{n-1}\sum_{i=1}^{n}\prod_{k=1}^{2}(X_i^{(k)} - \bar{X}_n^{(k)})\right), \qquad \text{(realized pseudo-volatility-cross)}$$

$$\hat{\sigma}_{(S)}(n;T_s,T_e) = \sqrt{\frac{n}{T_e - T_s}\left(\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2\right)}, \qquad \text{(realized pseudo-volatility)}$$

$$\hat{\rho}_{(S^{(1)},S^{(2)})}(n;T_s,T_e) = \frac{\sum_{i=1}^{n}\prod_{k=1}^{2}(X_i^{(k)} - \bar{X}_n^{(k)})}{\prod_{k=1}^{2}\sqrt{\sum_{i=1}^{n}\left(X_i^{(k)} - \bar{X}_n^{(k)}\right)^2}}. \qquad \text{(realized pseudo-correlation)}$$

Based on these pseudo-statistics, we can define the swaps, which are really forward contracts, by their payoffs at maturity date $T \geq T_e$:

$$\begin{aligned}
V_{\text{var}}(T) &= \alpha_{\text{var}} \cdot I \cdot \left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e) - \Sigma_K^2\right], && \text{(pseudo-variance swap)}\\
V_{\text{cov}}(T) &= \alpha_{\text{var}} \cdot I \cdot \left[\hat{\Sigma}^2_{(S^{(1)},S^{(2)})}(n;T_s,T_e) - \Sigma_K^2\right], && \text{(pseudo-covariance swap)}\\
V_{\text{vol}}(T) &= \alpha_{\text{vol}} \cdot I \cdot \left[\hat{\sigma}_{(S)}(n;T_s,T_e) - \sigma_K\right], && \text{(pseudo-volatility swap)}\\
V_{\text{corr}}(T) &= \alpha_{\text{corr}} \cdot I \cdot \left[\hat{\rho}_{(S^{(1)},S^{(2)})}(n;T_s,T_e) - \rho_K\right]. && \text{(pseudo-correlation swap)}
\end{aligned}$$

In the above, α_i are the converting parameters, $I = \pm 1$ is a long-short index, and Σ_K^2, σ_K and ρ_K are the strikes.

The problem is to price the above swaps. The general pricing formula is the mathematical expectation of the N_t-discounted claim under the unique martingale measure Q:

$$\begin{aligned}
V_i(0) &= \mathrm{E}^Q\left[\frac{N_0}{N_T}V_i(T)\right]\\
&= \mathrm{E}^Q\left[\mathrm{df}(0,T)V_i(T)\right]\\
&= \mathrm{df}(0,T)\,\mathrm{E}^Q\left[V_i(T)\right], \qquad i = \text{var, cov, vol, and corr.}
\end{aligned}$$

Note that the payoffs are path-dependent. Our goal is to find analytic expressions for the above pricing problems.

2.3 Price Pseudo-Variance & Pseudo-Covariance Swaps

To find the price of a pseudo-variance swap, it is necessary to compute the expected value of $\hat{\Sigma}^2_{(S)}(n;T_S,T_e)$. First, notice that each X_i is independently and normally distributed with mean

$$\alpha_i = \int_{t_{i-1}}^{t_i}\left(\mu_t - \frac{\sigma_t^2}{2}\right)dt,$$

and variance

$$\beta_i^2 = \int_{t_{i-1}}^{t_i} \sigma_t^2 dt.$$

Now we compute

$$\begin{aligned}
\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right] &= \mathrm{E}^Q\left[\frac{n}{T_e-T_s}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right)\right] \\
&= \frac{n}{T_e-T_s}\cdot\frac{1}{n-1}\,\mathrm{E}^Q\left[\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right] \\
&= \frac{n}{T_e-T_s}\cdot\frac{1}{n-1}\,\mathrm{E}^Q\left[\sum_{i=1}^{n}\left((\alpha_i+\beta_iY_i)-\frac{1}{n}\sum_{j=1}^{n}(\alpha_j+\beta_jY_j)\right)^2\right],
\end{aligned}$$

where Y_i is a normal $N(0,1)$ random variable. Let $\bar{\alpha} := \sum_{k=1}^{n}\alpha_i$ be the mean. After some calculations, the above expression can be written

$$\begin{aligned}
\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right] &= \frac{n}{T_e-T_s}\cdot\frac{1}{n-1}\left(\sum_{i=1}^{n}(\alpha_i-\bar{\alpha})^2+\mathrm{E}^Q\left[\sum_{i=1}^{n}\left(\beta_iY_i-\frac{1}{n}\sum_{j=1}^{n}\beta_jY_j\right)^2\right]\right) \\
&= \frac{n}{T_e-T_s}\cdot\frac{1}{n-1}\left(\sum_{i=1}^{n}(\alpha_i-\bar{\alpha})^2+\frac{n-1}{n}\sum_{i=1}^{n}\beta_i^2\right) \\
&= \frac{n}{T_e-T_s}\cdot\frac{1}{n-1}\sum_{i=1}^{n}(\alpha_i-\bar{\alpha})^2+\frac{1}{T_e-T_s}\sum_{i=1}^{n}\beta_i^2 \\
&= \frac{n}{T_e-T_s}\cdot\frac{1}{n-1}\sum_{i=1}^{n}(\alpha_i-\bar{\alpha})^2+\frac{1}{T_e-T_s}\int_{T_s}^{T_e}\sigma_\tau^2 d\tau \\
&= \frac{n}{T_e-T_s}\cdot\frac{1}{n-1}\sum_{i=1}^{n}(\alpha_i-\bar{\alpha})^2+\Sigma^2_{(S)}(T_s,T_e).
\end{aligned}$$

We point out that a variance swap can be replicated by a portfolio of options, forwards and zero-coupon bonds [2].

As pointed out in the problem statement, the pricing of a pseudo-covariance swap can be reduced to the pricing of a pseudo-variance swap. More specifically, it was argued that the price of a covariance swap can be written

$$V_{\mathrm{cov}}(0) = \frac{1}{4}\alpha_{cov}\cdot I\cdot \mathrm{df}(0,T)\cdot\left(\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S^{(1)}\cdot S^{(2)})}(n;T_S,T_e)\right]-\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S^{(1)}/S^{(2)})}(n;T_S,T_e)\right]-4\Sigma_K^2\right).$$

Hence we do not need to price the pseudo-covariance swap directly.

2.4 Price Pseudo-Volatility Swaps

To price the pseudo-volatility swap of strike σ_K, we need to compute the expected value of the realized-pseudo volatility $\hat{\sigma}_{(S)}(n;T_s,T_e)$. To get a second-order approximation, we use a Taylor expansion of the square root function about some $\hat{\Sigma}_0$ close to the expected swap value:

$$\begin{aligned}
\mathrm{E}^Q\left[\hat{\sigma}_{(S)}(n;T_s,T_e)\right] &= \mathrm{E}^Q\left[\sqrt{\hat{\Sigma}^2_{(S)}(n;T_S,T_e)}\right] \\
&\approx \mathrm{E}^Q\left[\frac{\hat{\Sigma}^2_{(S)}(n;T_S,T_e)+\hat{\Sigma}_0}{2\sqrt{\hat{\Sigma}_0}} - \frac{(\hat{\Sigma}^2_{(S)}(n;T_S,T_e)-\hat{\Sigma}_0)^2}{8(\hat{\Sigma}_0)^{\frac{3}{2}}}\right] \\
&= \frac{\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right]+\hat{\Sigma}_0}{2\sqrt{\hat{\Sigma}_0}} - \frac{\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)-\hat{\Sigma}_0\right]^2}{8(\hat{\Sigma}_0)^{\frac{3}{2}}}
\end{aligned}$$

$$= \frac{\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right]+\hat{\Sigma}_0}{2\sqrt{\hat{\Sigma}_0}} - \frac{\mathrm{Var}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right]+\left(\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right]-\hat{\Sigma}_0\right)^2}{8(\hat{\Sigma}_0)^{\frac{3}{2}}}.$$

Note that the expected value $\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right]$ is already computed in the previous section, so here we compute the variance:

$$\begin{aligned}
\mathrm{Var}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right] &= \mathrm{Var}^Q\left[\frac{n}{T_e-T_s}\left(\frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2\right)\right] \\
&= \frac{1}{(T_e-T_s)^2}\cdot\frac{n^2}{(n-1)^2}\mathrm{Var}^Q\left[\sum_{i=1}^n (X_i-\bar{X}_n)^2\right] \\
&= \frac{1}{(T_e-T_s)^2}\cdot\frac{n^2}{(n-1)^2}\left(\mathrm{E}^Q\left[\sum_{i=1}^n (X_i-\bar{X})^2\right]^2 - \left(\mathrm{E}^Q\left[\sum_{i=1}^n (X_i-\bar{X})^2\right]\right)^2\right).
\end{aligned}$$

After some calculations, the first term in the bracket above can be written

$$\mathrm{E}^Q\left[\sum_{i=1}^n (X_i-\bar{X})^2\right]^2 = \mathrm{E}^Q\left[\sum_{i,j=1}^n X_i^2X_j^2\right] - \frac{2}{n}\mathrm{E}^Q\left[\left(\sum_{k=1}^n X_k^2\right)^2\sum_{i=1}^n X_i^2\right] + \frac{1}{n^2}\mathrm{E}^Q\left[\sum_{i=1}^n X_i\right]^4.$$

Now, let us denote the coefficients by

$$\begin{aligned}
A &:= \mathrm{E}^Q\left[\sum_{i,j=1}^n X_i^2X_j^2\right], \\
B &:= \mathrm{E}^Q\left[\left(\sum_{k=1}^n X_k^2\right)^2\sum_{i=1}^n X_i^2\right], \text{ and} \\
C &:= \mathrm{E}^Q\left[\sum_{i=1}^n X_i\right]^4.
\end{aligned}$$

Then our formula for the variance can be written

$$\mathrm{Var}^Q\left[\sum_{i=1}^n (X_i-\bar{X})^2\right] = A - \frac{2}{n}B + \frac{1}{n^2}C - \left(\mathrm{E}^Q\left[\sum_{i=1}^n (X_i-\bar{X})^2\right]\right)^2.$$

Note that the last term has already been calculated in the previous section, so it remains to find formulas for A, B, and C. Indeed, we spent much of our time to do this and here are the results:

$$\begin{aligned}
A =& \sum_i (\alpha_i^4 + 6\alpha_i^2\beta_i^2 + 3\beta_i^4) + 2\sum_{i<j}(\alpha_i^2 + \beta_i^2)(\alpha_j^2 + \beta_j^2), \\
B =& \sum_i (\alpha_i^4 + 6\alpha_i^2\beta_i^2 + 3\beta_i^4) + 2\sum_{i<j}(\alpha_i^2 + \beta_i^2)(\alpha_j^2 + \beta_j^2) + 2\sum_{i<j}(3\alpha_i\beta_i^2 + \alpha_i^3)\alpha_j \\
&+ 2\sum_{i<j}\alpha_i(3\alpha_j\beta_j^2 + \alpha_j^3) + 2\sum_{i<j<k}\alpha_i\alpha_j(\alpha_k^2 + \beta_k^2), \\
C =& \sum_i (\alpha_i^4 + 6\alpha_i^2\beta_i^2 + 3\beta_i^4) + 4\sum_{i<j}(\alpha_i^3 + 3\alpha_i\beta_i^2)\alpha_j + 6\sum_{i<j}(\alpha_i^2 + \beta_i^2)(\alpha_j^2 + \beta_j^2) \\
&+ 12\sum_{i<j<k}(\alpha_i^2 + \beta_i^2)\alpha_j\alpha_k + 4\sum_{i<j}\alpha_i(\alpha_j^3 + 3\alpha_j\beta_j^2) + 12\sum_{i<j<l}\alpha_j(\alpha_j^2 + \beta_j^2)\alpha_l \\
&+ 12\sum_{i<j<l}\alpha_i\alpha_j(\alpha_l^2 + \beta_l^2) + 24\sum_{i<j<k<l}\alpha_i\alpha_j\alpha_k\alpha_l.
\end{aligned}$$

In the above, we have again used the notation that each random variable X_i follows a normal distribution with mean α_i and variance β_i^2.

Further, it can be shown that

$$\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n; T_S, T_e)\right] \to \Sigma^2_{(S)} \qquad \text{and} \qquad \mathrm{Var}^Q\left[\hat{\Sigma}^2_{(S)}(n; T_S, T_e)\right] \to 0,$$

as $n \to \infty$. This means that $\hat{\Sigma}^2_{(S)}(n; T_S, T_e)$ is an asymptotically unbiased and consistent estimator of $\Sigma^2_{(S)}(T_s, T_e)$.

2.5 Numerical Simulation

Following the idea of [1], we consider for the numerical simulations, a stochastic volatility model of the CIR type:

$$d\sigma_t^2 = \kappa(\theta^2 - \sigma_t^2)\,dt + \gamma\sigma_t\,dW_t^{(2)}. \tag{2.1}$$

Similar to the previous section, once discretized in time, X_t becomes a stochastic process taking random values, X_i, normally distributed with mean α_i and variance β_i^2, at each time step:

$$X_i = \log\left(\frac{S_i}{S_{i-1}}\right) \sim N(\alpha_i, \beta_i^2).$$

The parameters are related to the X_t process through the following equations:

$$\begin{aligned}
\alpha_i &= \mathrm{E}^Q[X_i] = \int_{t_{i-1}}^{t_i}\mu_t dt - \frac{1}{2}\mathrm{E}^Q\left[\int_{t_{i-1}}^{t_i}\sigma_t^2 dt\right], \\
\beta_i^2 &= \mathrm{Var}^Q[X_i] = \mathrm{E}^Q\left[\int_{t_{i-1}}^{t_i}\sigma_t^2 dt\right].
\end{aligned}$$

In order to evaluate the expectation and the variance of the X_t process, two different ways are being used. The first one is the application of the Monte-Carlo technique directly on the discretized σ_t^2 process, whose behaviour is given by (2.1). The second one is the application of the Monte-Carlo technique on the discretized process $X_i = \log(S_i/S_{i-1})$.

The advantage of using the CIR model is that it yields an exact solution for the expectation and the variance of the X_t process (cf.[1]). Hence, the numerical results can be easily verified.

All simulations are generated over 1000 time steps. For the results in Table 2.5, only 100 scenarios are considered, since the idea of the simulations is more to give an illustration and improve our understanding of the processes than to provide a numerical proof. The values of the parameters are set to $\kappa = 10$, $\theta^2 = 0.2$, $\sigma_0 = 0.2$ and $\gamma = 0.75$. Note that $\gamma = 0$ yields a deterministic volatility (see Figure 2.5).

	Expectation	Variance
Exact	0.1281	0.0019
MC σ_t^2	0.1287	0.0022
MC $(\log S_t)^2$	0.1280	0.0021

Table 2.1: Exact and numerical results for the expectation and variance of X_t.

Although the relatively low number of scenarios, the numerical results match the exact solutions within some reasonable error range.

2.6 Another Approach

Here we describe another approach in finding the expectation and variance. The idea is to work with the bond-discounted rate process S_t^* in the martingale measure Q. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space with $\mathcal{F}_t$ being filtration. We define the discounted S_t process by

$$S_t^* = \frac{S_t}{B_t},$$

where B_t is the deterministic bond price at time t satisfying $dB_t = r_t B_t \, dt$. It is known that the unique martingale measure Q for the discounted rate process S_t^* is given by the Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp\left(\int_0^T \frac{r_t - \mu_t}{\sigma_t} dW_t - \frac{1}{2} \int_0^T \left(\frac{r_t - \mu_t}{\sigma_t} \right)^2 dt \right). \tag{2.2}$$

Under this martingale measure, the driftless process S_t^* satisfies $dS_t^* = \sigma_t S_t^* \, dW_t^*$, where W_t^* given by

$$W_t^* = W_t - \int_0^t \frac{r_s - \mu_s}{\sigma_s} \, ds, \qquad 0 \le t \le T$$

is a standard Wiener process on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$. We also have

$$d \log S_t^* = -\frac{\sigma_t^2}{2} \, dt + \sigma_t \, dW_t^*.$$

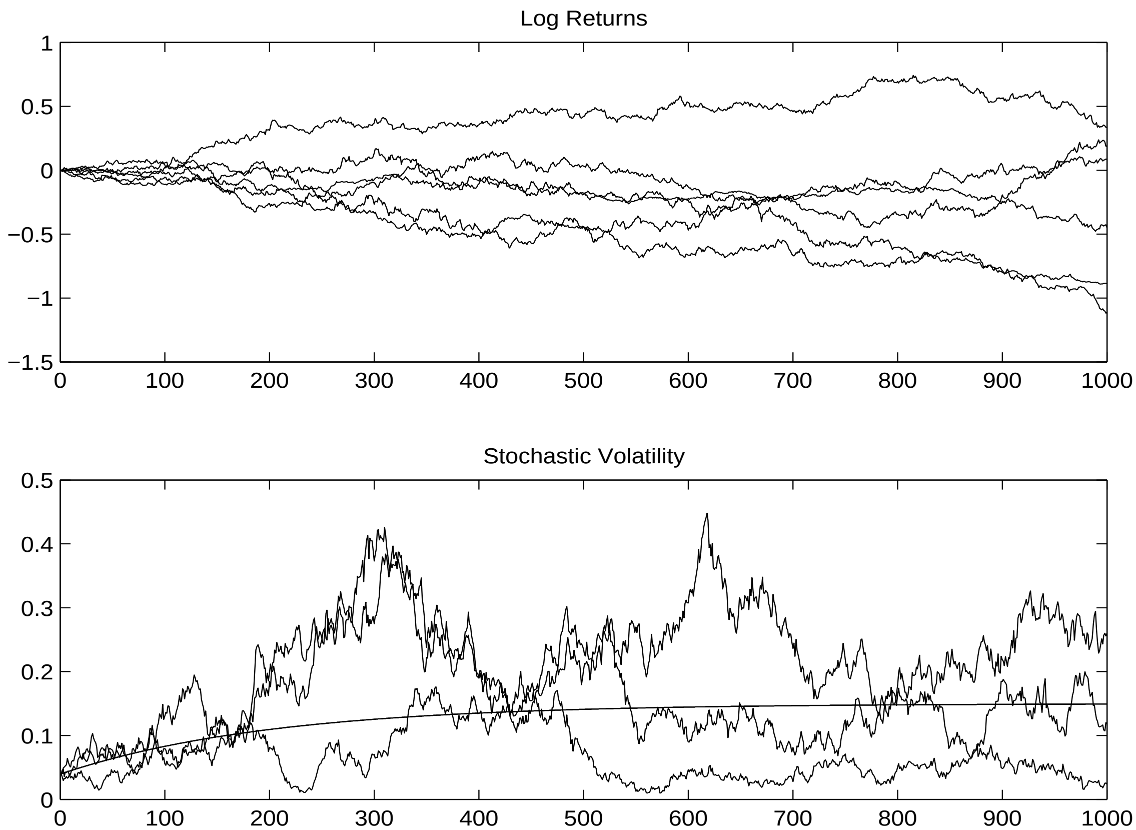

Figure 2.1: Monte-Carlo simulation, for the log-returns X_t and the stochastic volatility σ_t^2.

Then, the log-return for the discounted process S_t^* can be written

$$X_i^* := \log \frac{S_{t_i}^*}{S_{t_{i-1}}^*} = -\frac{1}{2} \int_{t_{i-1}}^{t_i} \sigma_t^2 \, dt + \int_{t_{i-1}}^{t_i} \sigma_t \, dW^*,$$

and it follows a normal distribution with mean

$$\alpha_i := \mathrm{E}^Q \left[X_i^* \right] = -\frac{1}{2} \int_{t_{i-1}}^{t_i} \sigma_t^2 \, dt$$

and variance

$$\beta_i^2 := \mathrm{Var}^Q \left[X_i^* \right] = \int_{t_{i-1}}^{t_i} \sigma_t^2 \, dt.$$

Thus, X_i^* has the representation $X_i^* = \alpha_i + \beta_i Y$, where Y is a standard normal $N(0,1)$. We note that

$$\mathrm{E}^Q \left[(X_i^*)^2 \right] = \alpha_i^2 + \beta_i^2 \qquad \text{and} \qquad \mathrm{E}^Q \left[(X_i^*)^4 \right] = \alpha_k^4 + 6\alpha_i^2 \beta_i^2 + 3\beta_i^4.$$

Now, we compute the expected value of the pseudo-volatility-square, first neglecting the mean $\bar{X}_n^* := \frac{1}{n} \sum_{i=1}^n X_i^*$:

$$\mathrm{E}^Q \left[\hat{\Sigma}_{(S)}^2(n; T_S, T_e) \right] = \mathrm{E}^P \left[\hat{\Sigma}_{(S)}^2(n; T_S, T_e) M_T \right].$$

Here M_T is defined in the right-hand side of (2.2). If we use Taylor expansion for M_T, then we obtain the formula demonstrated below. We propose to calculate E^Q for the discounted process S_t^*, which simplifies some calculation in the case of S_t:

$$\begin{aligned}\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S^*)}(n;T_S,T_e)\right] &= \frac{n}{(T_e-T_s)}\cdot\frac{1}{(n-1)}\sum_{i=1}^{n}\mathrm{E}^Q\left[(X_i^*)^2\right]\\ &= \frac{n}{(T_e-T_s)}\cdot\frac{1}{(n-1)}\sum_{i=1}^{n}\left(\alpha_i^2+\beta_i^2\right)\\ &= \frac{n}{(T_e-T_s)}\cdot\frac{1}{(n-1)}\sum_{i=1}^{n}\left(\frac{1}{4}\beta_i^4+\beta_i^2\right).\end{aligned}$$

If we do not neglect $\bar{X}_n^*$, then we obtain the more general expression

$$\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right] = \frac{n}{(T_e-T_s)}\cdot\frac{1}{(n-1)}\sum_{i=1}^{n}\left[\left(-\frac{1}{2}\beta_i^2+\frac{1}{2n}\sum_{j=1}^{n}\beta_j^2\right)^2+\frac{n-1}{n}\beta_i^2\right].$$

In the risk-neutral world, the process S_t satisfies the SDE

$$d\log S_t = \left(r_t-\frac{\sigma_t^2}{2}\right)dt+\sigma_t\,dW_t^*.$$

This means that under the measure Q the log-return X_i is normally distributed with mean

$$\alpha_i = \int_{t_{i-1}}^{t_i}\left(r_t-\frac{\sigma_t^2}{2}\right)dt$$

and variance

$$\beta_i^2 = \int_{t_{i-1}}^{t_i}\sigma_t^2\,dt.$$

In this case our formula becomes

$$\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right] = \frac{n}{(T_e-T_s)}\cdot\frac{1}{n-1}\sum_{i=1}^{n}\left[\left(\alpha_i-\frac{1}{n}\sum_{j=1}^{n}\alpha_j\right)^2+\frac{n-1}{n}\beta_i^2\right],$$

which is the same as in Section 2.3.

To compute the variance, we need to find the the following expectation:

$$\begin{aligned}\mathrm{E}^Q\left[\left(\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right)^2\right] &= \frac{1}{(T_e-T_s)^2}\cdot\frac{n^2}{(n-1)^2}\left(\sum_{i=1}^{n}\mathrm{E}^Q\left[(X_i^*)^4\right]+2\sum_{i<j}\mathrm{E}^Q\left[(X_i^*)^2(X_j^*)^2\right]\right)\\ &= \frac{1}{(T_e-T_s)^2}\cdot\frac{n^2}{(n-1)^2}\left(\sum_{i=1}^{n}(\alpha_i^4+6\alpha_i^2\beta_i^2+3\beta_i^4)\right.\\ &\qquad\left.+2\sum_{i<j}\left(\alpha_i^2\alpha_j^2+\alpha_i^2\beta_j^2+2\beta_i^2\alpha_j^2+3\beta_i^2\beta_j^2+2\alpha_i\alpha_j\beta_i\beta_j\right)\right).\end{aligned}$$

Since $\alpha_i=-\frac{1}{2}\beta_i^2$, the above expression can be simplified.

2.7 Conclusion

During the short week, our group has obtained closed-form formulas for the expectation and variance of the realized pseudo-volatility-square, as requested by the problem:

$$\mathrm{E}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right] \qquad \text{(Question 1)}$$
$$\mathrm{Var}^Q\left[\hat{\Sigma}^2_{(S)}(n;T_S,T_e)\right] \qquad \text{(Question 2)}$$

Although these formulas are complicated, they can be (carefully) implemented on a computer to compute prices of pseudo-variance, pseudo-covariance, and pseudo-volatility swaps in real time. This is the advantage of formulas over (quasi-) Monte-Carlo simulation and is highly desired by market practitioners.

In theory, we should be able to obtain the answer for question three as well, that is, to compute the covariance of different combinations of

$$\begin{aligned} A_1(n) &:= \hat{\Sigma}^2_{(S^{(1)})}(n;T_S,T_e), \\ A_2(n) &:= \hat{\Sigma}^2_{(S^{(2)})}(n;T_S,T_e), \\ A_{12}(n) &:= \hat{\Sigma}^2_{(S^{(1)}\cdot S^{(2)})}(n;T_S,T_e), \qquad \text{and} \\ A_{1/2}(n) &:= \hat{\Sigma}^2_{(S^{(1)}/S^{(2)})}(n;T_S,T_e). \end{aligned}$$

However, we did not have time to complete this task, so this can be part of future work. Also, we did not have the opportunity to verify the appropriateness or validity of our results with our industry representative Ritchie He of RBC Financial Group, since he was away for the last days of the week.

Bibliography

[1] Brockhaus, O. and Long, D., Volatility Swaps Made Simple, *Risk*, January 2000, pp. 92–95.

[2] Demeterfi, K., Derman E., Kamal, M., and Zou J., A Guide to Variance Swaps, *Risk*, June 1999, pp. 54–59.

[3] Demeterfi, K., Derman E., Kamal, M., and Zou J., A Guide to Volatility and Variance Swaps, *The Journal of Derivatives*, Summer 1999, pp. 9–32.

[4] Hunt, P.J. and Kennedy, J.E., *Financial Derivatives in Theory and Practice*, John Wiley & Sons, 2000.

[5] Hunt, P.J., Kennedy, J.E., and Scott, E.M., Terminal Swap-Rate Models, Working paper, University of Warwick, 1996.

[6] Musiela, M., Rutkowski, M., *Martingale Methods in Financial Models*, Springer-Verlag, 1997.