Chapter 1

General Statistical Design of an Experimental Problem for Harmonics

Problem Presented By: Bill Mawby (Michelin Tire Corporation)

Mentors: Sean Bohun (Penn State Mont Alto); Peter Gibson (York University); Michael Lamoureux (University of Calgary)

Student Participants: Benjamin Akers (University of Wisconsin - Madison); Andreas Hofinger (Johannes Kepler University, Linz); Jason Lobb (Carleton University); Malcolm Roberts (University of Alberta)

Report prepared by: Michael Lamoureux (`mikel@math.ucalgary.ca`)

1.1 Introduction

Each year, the Pacific Institute for the Mathematical Sciences organizes a one-week long Industrial Problem Solving Workshop (IPSW), where researchers from academia and industry work for five intense days to make progress on solving specific mathematical problems that arise in real industrial environments. Typically, many new, interesting mathematical results are obtained that provide concrete value to the companies proposing the problems.

Four years ago, the Michelin Tire Corporation proposed a problem on experimental design, to improve the manufacturing process for their tires. The idea is basically to determine the effects of placements for various layers built up in the construction of a tire, to allow the design of a smooth tire with a smooth ride. A highly success solution was developed, and it has been reported that this method introduced savings of over half a million dollars in their test processes. This year, Michelin returned to the workshop with an extension to the original problem, to address specific refinements in the testing method. This report summarizes the work completed in course of the five day workshop.

1.2 Problem Description

Tires are subjected to a variety of force measurements that are stored as periodic waveforms. Harmonic components of these waveforms are related to tire performance characteristics such as noise and comfort and hence the control and reduction of the amplitudes of these harmonics is an important activity of manufacturing. Technicians may choose to perform designed experiments on their production processes to understand better their impact on the resulting force harmonics. It could be advantageous to have a general design of experiment methodology which allows technicians to choose optimal designs for their studies.

To make this more concrete consider two types of forces (F and G). F is characterized by 5 harmonics F1-F5 and G is characterized by 10 harmonics (G1-G10). Practically, the technician might have 20 different process elements (P1-P20) that can be rotated within the construction of the tire and which can affect the force measurements. It is assumed that rotation of a production process will result in the equivalent rotation of the force measurement and that superposition of P1 through P20 will result in a corresponding superposition of resulting Fi's. In general the movement of any process element such as P1 may affect all harmonics and forces (F1-F5 and G1-G10). The general problem is to choose the angles of rotation for a set of Ps so that the harmonic effects are well estimated and the cost of experimentation is minimized. Note that the variance of the estimates is related to the angles chosen (for example choosing 180 degrees prevents the estimation of the even harmonics) and that the cost of a study is proportional to the number of angles that are used in the design. Other features of interest include the reparability/extensibility of designs, identifying sets of competitors/surrogates and allowing different precision for different harmonics.

The original problem in 2000 was proposed as follows: Develop a method that allows estimation of n harmonics on m production steps from sampled waveforms on tires. This method should be flexible, robust and easily constrained to meet operating conditions. For example we measure a waveform of force variation on a cured tire sampled at 256 equally-spaced points made with 5 products with joints at fixed angular positions. Then we change these relative angular positions, construct a new tire and measure its waveform. We would like to decompose the overall effect represented in each tire into contributions due to each product. We can do this with each harmonic of the waveform but would like to ensure good estimation of all effects for all of a specified set of harmonics.

The new problem for 2004 would be to extend the previous PIMS IPSW 2000 results on Statistical Design in any of several directions that are discussed below.

1. Develop fully the method called the Good Lattice Points (GLP) method presented in the proceedings [1] so that it could be implemented in practice to allow estimation without the prime number restriction.

2. Include the fitting of a few non-harmonic frequencies with the harmonics and find good designs for these (assumes that all harmonics are not fitted) such as a frequency that passes through the signal exactly 0.62 times. The harmonics are relative to the tire circumference in the old problem, but sometimes effects such as extrusion or measuring devices put sinusoidal patterns into the overall waveform but these effects are not harmonics (the periods are not integral divisors of the tire circumference) of the tire but rather have periods that are fractional parts (like 0.62). We use multiple linear regression to estimate non-harmonic frequency effects and harmonic frequency effects simultaneously (with some correlation between the estimates). The problem is to provide

an optimal design strategy for this situation given that we can provide some information like number of non-harmonic frequency effects and possible ranges for their frequencies.

3. Determine the best design for a function of the harmonics of different types such as $f(\text{type 1 harmonic 1}, \text{type 2 harmonic 1})$ where the experiment is performed on rotation of tire components as before and f is of a specified class or form. In this case we measure two or more type of waveforms on each tire as before. We then combine these multiple outputs into single derived output (often linearly by summation etc. but it could be non-linear). We want to ensure adequate estimation of the product effects for each of a selected set of harmonics for this derived output.

4. Expand the concept to a two dimensional Fourier transform or equivalent where the surface could be considered flat or as the surface of an inflated tire (semi-toroidal).

1.3 Methodology

It was clear early in the workshop that this problem could be handled quickly by reviewing the analysis which was done in 2000, and extending those ideas to the new problems at hand. We reviewed the required Fourier techniques to describe the harmonic problem, and statistical techniques to deal with the linear model that described how to accurately measure quantities that come from real experimental measurements. The "prime method" and "good lattice points method" were reviewed and re-analysed so we could understand (and prove) why they work so well. We then looked at extending these methods and successfully found solutions to problem 1) and 2) above. Matlab code was written to test and verify the algorithms developed. We have some ideas on problems 3) and 4), which are described below.

1.4 Fourier Series Formulation - a review

The tire's force characteristic curve can be described as a continuous function on the interval $[0, 1]$ with periodic boundary conditions. As the problem was originally posed, it was suggested that these periodic functions $f \in C[0, 1]$ be expanded in terms of sines and cosine functions: that is, one writes

$$f(t) = \sum_{n=0}^{\infty} \left[a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right],$$

where a_n, b_n are real coefficients encoding the magnitude and phase of the corresponding harmonic. Algebraically, it is more convenient to use complex exponentials to expand the periodic function in a Fourier series, as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}, \qquad c_n = \int_0^1 f(t)e^{-2\pi int}\,dt.$$

Since f is real, $\bar{f} = f$ with the bar denoting the complex conjugate. As a result, the c_n must satisfy the condition

$$\bar{c}_n = \int_0^1 \bar{f}(t)e^{2\pi int}\,dt = \int_0^1 f(t)e^{2\pi int}\,dt = c_{-n}.$$

More important is the redundancy due to the fact that the continuous function $f \in C[0,1]$ is observed only at finitely many equally spaced points $t_k = k/2N$, for $k = 0, 1, \ldots, 2N-1$ and we can represent f as

$$f(t) = \sum_{k=0}^{2N-1} f(t_k)\delta(t - t_k).$$

Using this representation the Fourier coefficients take the form

$$c_n = \sum_{k=0}^{2N-1} f(t_k)e^{-2\pi i n t_k}$$

and as a result

$$c_{n+2N} = \sum_{k=0}^{2N-1} f(t_k)e^{-2\pi i(n+2N)k/2N} = \sum_{k=0}^{2N-1} f(t_k)e^{-2\pi i n t_k}e^{-2\pi i k} = c_n.$$

We see that observing f at only finitely many points introduces a redundancy in the Fourier coefficients effectively collapsing the Fourier series expansion to the finite sum

$$f(t_k) = \sum_{n=-N+1}^{N} c_n e^{2\pi i n t_k}.$$

It is worth noting in passing that the Fourier coefficients c_n are quickly calculated using an FFT software routine, and so this formulation of the problem does not introduce any additional complexity in the problem.

In the problem at hand, a function f is considered the signature of a given tire component the will affect the final force profile. If this component is rotated by an angle θ, the signature function f is shifted and the corresponding Fourier coefficients change. Introducting the notation S_θ for the shift operator, one obtains

$$(S_\theta f)(t) = f(t - \theta) = \sum_n c_n e^{2\pi i n(t-\theta)} = \sum_n (e^{-2\pi i n\theta} c_n)e^{2\pi i n t} = \sum_n c_n^\theta e^{2\pi i n t}.$$

That is, the Fourier coefficients transform under the shift by θ as a linear transform $c_n \mapsto e^{-2\pi i n\theta} c_n$. Equivalently, the vector of coefficients c_n transforms as

$$\begin{pmatrix} \vdots \\ c_n \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ c_n^\theta \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & & \\ & e^{-2\pi i n\theta} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ c_n \\ \vdots \end{pmatrix}$$

which is just multiplication by a diagonal matrix D_θ whose entries are complex exponentials.

1.5 The Component Problem - a review

A tire is built up from a number of layers (tread, cords, airtight inner rubber, etc.), usually on the order of $m = 20$ components. Each layer contributes a signature $f^k(t) \in C[0,1]$ to the observed force profile

$$F(t) = \sum_{k=1}^{m} f^k(t),$$

where the assumption (based on Michelin's experience) is that the contribution is additive. The component functions f^k cannot be measured directly; however, a factory technician may modify the construction of the tire by changing the positioning of individual layers within the test tire. Each layer may be shifted independently by some angle θ. Applying a vector of shifts $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$, where θ_k is the rotation angle for kth layer, gives an operation on the observed force profile as

$$(S_\Theta F)(t) = \sum_{k=1}^{m} (S_{\theta_k} f^k)(t).$$

In Fourier components, this becomes

$$\sum_n C_n^\Theta e^{2\pi int} = \sum_k \sum_n \left(e^{-2\pi in\theta_k} c_n^k\right) e^{2\pi int} = \sum_n \left(\sum_k e^{-2\pi in\theta_k} c_n^k \right) e^{2\pi int},$$

and by equating terms in the Fourier expansion one obtains the transform directly on the coefficients as

$$C_n^\Theta = \sum_{k=1}^{m} e^{-2\pi in\theta_k} c_n^k.$$

In particular, one observes there is no mixing of harmonics: that is, the nth harmonic of the observed (transformed) force curve is a weighted sum of the nth harmonics of the contributing layers.

The analysis problem is to determine the coefficients c_n^k from the observed C_n^Θ, using some choice of the vector of angles Θ. Since the observed spectra are real, it is enough to consider only non-negative n in determining the harmonics, and the constant term ($n = 0$) is irrelevant. In practice, only a small number of low harmonics is of interest (eg. $n = 1, 2, \ldots, 5$), but the coefficients must be determined for all layers (eg. $k = 1, 2, \ldots, 20$). Also note this is becomes a statistical problem, as the measured coefficients include measurement error and statistical deviations due to variations in the construction of these real tires.

1.6 The Linear Model - a review

The problem is to determine individual coefficients c_n^k, for all layers $k = 1, \ldots, m$, from observations of the lumped coefficients C_n^Θ, where the experimental design involves choosing some appropriate vectors of angles $\Theta = (\theta_1, \theta_2, \ldots, \theta_m)$. The design also should find how many vectors $(\Theta^1, \Theta^2, \ldots, \Theta^R)$ are required to accurately determine the coefficients c_n^k. These coefficients must be determined for a range of harmonics, say $n = 1, \ldots, q$, and it will be convenient to design the experiments to work for all these harmonics simultaneously.

It is natural to group the coefficients into vectors, as

$$\boldsymbol{c}^k = \begin{pmatrix} c_1^k \\ \vdots \\ c_q^k \end{pmatrix}, \qquad \boldsymbol{C}^\Theta = \begin{pmatrix} C_1^\Theta \\ \vdots \\ C_q^\Theta \end{pmatrix}.$$

The linear model encompassing all layers, and the range of harmonics, can thus be written in block form as

$$\begin{pmatrix} \boldsymbol{C}^{\Theta^1} \\ \boldsymbol{C}^{\Theta^2} \\ \vdots \\ \boldsymbol{C}^{\Theta^R} \end{pmatrix} = \begin{pmatrix} D_{\theta_1^1} & D_{\theta_2^1} & \dots & D_{\theta_m^1} \\ D_{\theta_1^2} & D_{\theta_2^2} & \dots & D_{\theta_m^2} \\ \vdots & & & \vdots \\ D_{\theta_1^R} & D_{\theta_2^R} & \dots & D_{\theta_m^R} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}^1 \\ \boldsymbol{c}^2 \\ \vdots \\ \boldsymbol{c}^m \end{pmatrix} + \boldsymbol{\epsilon}$$

where $\Theta^1, \dots, \Theta^R$ is the choice of vectors of angles set in the experiment, the $D_{\theta_k^j}$ are $q \times q$ diagonal matrices with entries $\exp(-2\pi i n \theta_k^j)$ on the diagonal, and $\boldsymbol{\epsilon}$ is the statistical measurement error.

More succinctly, the linear model is represented by $\boldsymbol{C}^{\boldsymbol{\Theta}} = D\boldsymbol{c} + \boldsymbol{\epsilon}$, with D a matrix in block form, each block a diagonal matrix as above. If these were real matrices, the solution via least squares is clear; it is a simple exercise to verify that even for complex matrices the least square solution (using complex inner products) is straightforward. Namely, one solves for $\boldsymbol{c}$ as

$$\hat{c} = (D^\dagger D)^{-1} D^\dagger \boldsymbol{C}^{\boldsymbol{\theta}},$$

where $D^\dagger$ indicates the complex conjugate transpose of the matrix D. Similarly, the variance estimates for the inversion will depend on the properties of matrix $(D^\dagger D)^{-1}$.

Noting that $D^\dagger D$ is also in block form, it is convenient to permute rows and columns (essentially grouping terms by layers, rather than harmonics) to obtain the matrix $X = \mathrm{Perm}(D)$ so that $X^\dagger X$ is in block diagonal form, with

$$X^\dagger X = \begin{pmatrix} (Z_1) & & & 0 \\ & (Z_2) & & \\ & & \ddots & \\ 0 & & & (Z_q) \end{pmatrix},$$

where each $m \times m$ block (Z_n) has entries

$$(Z_n)_{jk} = \sum_{r=1}^{R} e^{2\pi i n(\theta_k^r - \theta_j^r)}.$$

This greatly simplifies the analysis, since each block (Z_n) may be examined separately. Notice each such block corresponds to a separate harmonic.

The problem becomes that of estimating the regression coefficients in the multiple regression model

$$\boldsymbol{C}^{\boldsymbol{\Theta}} = X\boldsymbol{c} + \boldsymbol{\epsilon}$$

where for simplicity we have used $\boldsymbol{C}^{\boldsymbol{\Theta}}$, $\boldsymbol{c}$ and $\boldsymbol{\epsilon}$ to denote the reordered versions of these vectors. A standard assumption is that

$$\mathrm{Var}(\boldsymbol{\epsilon}) = \sigma^2 I_{qm}$$

for some (unknown) σ. It was pointed out that this assumption essentially says the R tires and q harmonics act independently, and different measurements have equal error; this may be a gross over-simplifcation worth further investigation. For instance, there may be some bias in the way the tires are constructed for the test, or trends reflected in the sequence that the tires are built. On the other hand, the harmonics are orthogonal measures in a large dimensional space, and at least some of us were convinced that the first few harmonics would act independently, with similar measurement error. In any case, we proceed with this assumption.

The least-square estimator of $\boldsymbol{c}$ is

$$\hat{c} = (X^\dagger X)^{-1} X^\dagger \boldsymbol{C}^{\boldsymbol{\Theta}}$$

with variance

$$\mathrm{Var}(\hat{c}) = (X^\dagger X)^{-1} X^\dagger \,\mathrm{Var}(\boldsymbol{\epsilon}) X (X^\dagger X)^{-1} = \sigma^2 (X^\dagger X)^{-1}.$$

Thus the equation of finding an optimal design boils down to finding a matrix X such that $X^\dagger X$ is "good".

Some possible optimality conditions ("goodness" of X) include minimizing the determinate of matrix $(X^\dagger X)^{-1}$ (D-optimality), minimizing the spectral norm of $(X^\dagger X)^{-1}$, or minimizing the maximum eigenvalue of $(X^\dagger X)^{-1}$. It turns out these three conditions are equivalent, since the matrix $X^\dagger X$ has a trace independent of the choice of angles (equal to mqR) and thus the minimum occurs when all eigenvalues are equal, and $X^\dagger X$ is R times the identity matrix. Generally speaking, the closer $X^\dagger X$ is to diagonal, the better.

Another optimality condition is to fix some vector $\boldsymbol{w}$ and minimize the variance $\mathrm{Var}(\boldsymbol{w}^{\mathrm{T}}\hat{c})$, which is a weighted sum of the entries of $\hat{c}$. This would be of interest to the manufacturer when some harmonics, or some layers, are deemed to be more important than others.

1.7 The Prime, GLP, and SLP Methods - a review

A key result of our earlier project in 2000 was the development of three methods to pick a matrix Z_n for the nth harmonic in such a way that the off-diagonal terms of the covariance matrix $Z_n^\dagger Z_n$ are all zero. These three methods were called, respectively, the Prime Method, the Good Lattice Point Method, and the Simplified Lattice Point Method. They all depended on the idea of picking the entries of matrix Z_n to be suitable (complex) roots of unity, in such a way that the corresponding sums of roots always came out to zero on the off-diagonal. Moreover, this optimal choice of design angles can be chosen in such a way that works for a range of harmonics.

As a simple example, for a design matrix of size $m = 5$, we use the prime method to choose design angles of the form

$$(\theta_k^j) = \frac{2\pi}{5} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}$$

and set the matrix entries for the nth harmonic matrix Z_n as

$$(Z_n)_{jk} = e^{in\theta_k^j}.$$

Indeed, we recognize matrix Z_1 as the matrix for a discrete Fourier transform of dimension 5, which is of course orthogonal, and hence $Z_1^\dagger Z_1$ is diagonal, as desired. Moreover, since $\mathbb{Z}/5$ is a field, it is easy to see that for the nth harmonic, the action of n on entries $n(\theta_k^j)$ just permutes the rows around ($n \neq 0 \bmod 5$), so this design works equally for all harmonics n which are not multiples of 5.

In general, for m equal to any prime, the Prime Method prescribes the design matrix of angles to be an $m \times m$ matrix with entries

$$\theta_k^j = \frac{2\pi}{m}\{(j-1)(k-1) \bmod m\}.$$

Again, this design gives optimal Z_n for any prime m and any harmonic n which is not a multiple of m.

Theorem 1 (THE PRIME METHOD). *For integer m prime, n not a multiple of m,and design angles chosen as*

$$\theta_k^j = \frac{2\pi}{m}\{(j-1)(k-1) \bmod m\},$$

then the corresponding design matrix Z_n with entries

$$(Z_n)_{jk} = e^{in\theta_k^j}$$

is optimal.

Proof. A proof is provided in reference [1], but it is easy enough to observe that matrix Z_n is a DFT matrix of size $m \times m$, with rows permutated around by multiplication by n. □

The Prime Method is a powerful method both in that it gives explicitly an optimal solution and it works for a range of harmonics. However, it has some disadvantages. It is somewhat inflexible in the number of layers m, as m must be prime. It requires using as many angles as there are layers. This can be an expensive, if not impossible, construction in some tire plants. In addition, there is the problem of setting angles exactly.

The second method is inspired by the Good Lattice Point (GLP) method of Fang and Wang [5], which uses a careful choice of lattice points in an m-dimensional hypercube to accelerate integration over a multidimensional Riemann sum. The basic insight is to look for angle combinations which will lead to sequences $\{\theta_k^r - \theta_j^r\}_{r=1}^R$ which will allow for fast convergence of the sum

$$\sum_{r=1}^{R} \frac{1}{R} e^{ni(\theta_k^r - \theta_j^r)} \to \int_0^1 e^{2\pi inx} dx.$$

With the Monte Carlo method, random sampling of the hypercube produces random sequences on $[0,1]$, but with a convergence of order $R^{-1/2}$. The GLP method will exhibit convergence at the faster rate of $R^{-1}\log^m(R)$. We tested the GLP method to see if we would obtain a good sequence of angles, and came up with a surprising conjecture.

First, recall the definition of a lattice point set and a GLP set. Let $(R; h_1, h_2, \ldots, h_m)$ be a vector of positive integers satisfying i) $m < R$, ii)$1 \leq h_j < R$, iii) $h_j \neq h_k$, for all $j \neq k$, iv) (h_j, R) are coprime, for all j.

The *lattice point set* of the generating vector $(R; h_1, h_2, \ldots, h_m)$ is the set of vectors $\{(x_{r1}, \cdots, x_{rm}), r = 1, \cdots, R\}$, with values

$$x_{rj} = \mathrm{frac}\left(\frac{2rh_j - 1}{2R}\right),$$

where "frac" denotes the fractional part of the given real number. If this set has the smallest discrepancy (defined in Fang and Wang), it is called a GLP set.

The principle behind GLP sets is that generating vectors can always be constructed so that a GLP set is created, whose points are uniformly distributed about the hypercube. Fang and Wang tabulate many different choices for a range of R and m, corresponding in our case to numbers of tires R in the experimental design, and number of layers m per tire.

In our tire example, the vector of angles are obtained from the lattice point sets by scaling by a factor of 2π, so $\theta_k^j = 2\pi x_{jk}$. We tested a number of the GLPs from the book to determine how close they are to being optimal. In every instance the GLP set was optimal. We have the following:

Conjecture 1 (GLP METHOD). *Every GLP set produces an optimal design. That is, for each harmonic n as above, the design matrix Z_n satisfies*

$$(Z_n^\dagger Z_n)^{-1} = \frac{1}{R} I_m$$

exactly. Moreover,

- *m can be chosen arbitrarily (not necessarily prime)*
- *R can be chosen arbitrarity (although prime is a popular choice)*
- *the same design is optimal for all harmonics co-prime with R.*

In the workshop, there was not enough time to explore how GLP sets were constructed in the literature, so it was not clear to us how optimal designs were resulting from these choices. A quick review of work in the area indicates some number theoretical results are being used to construct the charts of Fan and Wang. However, a simple examination shows the differences $x_{rj} - x_{rk} = \mathrm{frac}(r(h_j - h_k)/R)$, so as in the prime method, the sum in the covariance matrix will cycle around a subset of the R roots of unity. For a good choice of the h_j, this subset will always sum to zero. Thus, while this is short of a proof verifying the GLP method works, there is the basis for another useful technique, the Simplified Lattice Method.

The third method we developed in the 2000 workshop is called the Simplified Lattice Method, which is also inspired by the GLP method of Fang and Wang, but for which we could provide a proof of optimality.

Theorem 2 (SLP METHOD). *Fix an integer $m > 0$ and fix $\mathbf{N}$ a subset of $\{1, 2, \ldots\}$. Suppose $(R; h_1, h_2, \ldots, h_m)$ with $m < R$ is a vector of positive integers such that R is not a divisor of $n(h_j - h_k)$ for any $n \in \mathbf{N}, j \neq k$. Then the vectors of angles (scaled lattice points) defined by*

$$\theta_k^j = \frac{2\pi}{R}(j \cdot h_k \bmod R), \qquad 1 \leq j \leq R, 1 \leq k \leq m$$

gives an optimal design for all harmonics $n \in \mathbf{N}$. That is, the design matrix Z_n satisfies

$$(Z_n^\dagger Z_n)^{-1} = \frac{1}{R} I_m.$$

Proof. For harmonic n in the set $\mathbf{N}$ and $j \neq k$, the integer $n(h_j - h_k)$ is not divisible by R and hence the map $r \mapsto n(h_j - h_k)r \bmod R$ defines an endomorphism on the ring $\mathbb{Z}/R$ which has more than one element in its range, which is a subring of $\mathbb{Z}/R$. Thus when scaled by $2\pi i$ and exponentiated, one obtains some r roots of unity, for some divisor $r > 1$ of R. Thus the sum

$$\sum_{r=1}^{R} e^{2\pi n r(h_k - h_j)/R}$$

simply cycles around these r roots of unity, and so sum to zero. Hence the off-diagonal terms of the covariance matrix $X^\dagger X$ are zero, the diagonal terms are R, and thus the optimal design is achieved. □

These criteria are easy to fulfill in any situation of tires, as shown in the following.

Example 1. *With m the number of layers in the tire, and $\mathbf{N}$ a finite set of harmonics, let R be any prime number strictly bigger than m and all integers $n \in \mathbf{N}$. Then the integer vector $(R; 0, 1, 2, \ldots, m-1)$ generates lattice points yielding an optimal design.*

This example gives a method much like the original prime method described above. However, the number of layers m need not be prime, and one can select any finite set of harmonics, yet still obtain an optimal design. The number of tires R need not be prime: one could choose a composite number with some prime factor bigger than m and all n. There remains the disadvantage that almost every layer on almost every tire must be set to a non-zero angle, and the angles must be set to accuracies on the order of $2\pi/R$.

1.8 Problem 1: Extending the Simplified Lattice Method

All the methods developed above involve choosing design angles θ_k^j as simple fractions scaled by 2π, which then determine a matrix Z_n with entries that are roots of unity. During the current workshop, we experimented with a number of other choices for design angles and roots. For instance, we have this sense that distributing the roots randomly, but more or less uniformly around the unit circle might be a good idea. We tried taking multiples of an irrational for the design angles, say

$$\theta_k^j = j \cdot k \cdot \frac{1+\sqrt{5}}{2}$$

as multiples of the golden ratio, or multiples of $\sqrt{2}$, and so forth. The combinatorist in our group tried assorted permutations of roots of unity to try to obtain maximum orthogonal columns for the matrix.

However, our numerical tests suggested nothing we did was better than taking a simple DFT matrix, and modifying for harmonics. That is, for a given matrix size R, we pick design angles

$$\theta_k^j = 2\pi \frac{(j-1)(k-1)}{R}, \qquad 1 \leq j, k \leq R$$

which results in the nth harmonic matrix Z_n with entries

$$(Z_n)_{jk} = e^{in\theta_k^j}.$$

Thus Z_1 is the DFT matrix, and the other Z_n matrices have rows (and columns) which are just a permuted subset of those of Z_1. If it is a full subset, we obtain an orthogonal matrix, and hence an optimal design.

This full subset is obtained precisely when the action of n on the fractions $(j-1)(k-1)/R$ is just a permutation. A moments thought on divisors of R and n shows the following is true.

Theorem 3 (DFT MATRIX METHOD). *Fix an integer $R > 0$ and choose design angles*

$$\theta_k^j = 2\pi\frac{(j-1)(k-1)}{R}, \qquad 1 \le j,k \le R$$

and corresponding nth harmonic matrix Z_n with

$$(Z_n)_{jk} = e^{in\theta_k^j}.$$

Then the design matrix is optimal $((Z_n^\dagger Z_n)^{-1} = I/R)$ if and only if n and R are relatively prime.

So, for instance, for an experiment with $R = 20$ tires, and $m = 20$ layers under control, we obtain an optimal design precisely for those integers n that have no common divisors with 20. For instance, that would be

$$n = 1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, \dots .$$

In particular we can determine all the odd harmonics which are not a multiple of 5. Notice the harmonic need not be smaller than $R = 20$, just relatively prime to it.

As another example, with $R = 16$ tires, and $m = 16$ layers under control, we obtain optimal designs for harmonics

$$n = 1, 3, 5, 7, 9, \dots .$$

That is, for all odd harmonics (including those larger than 20.)

In the original problem, we were asked to design for harmonics $n = 1, 2, 3, 4, 5$. Trying to find a small number R that is relatively prime to these 5 numbers (simultaneously), and also small, typically gives R as a prime number. For instance, the possible R relatively prime to $1, \dots, 5$ are

$$R = 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, \dots$$

and we see the first non-prime solution (other than 1) is $R = 49$. This explains why the Good Lattice Point method we described above often degenerated into the Prime Method.

However, if one is willing to look for only a select set of harmonics (such as just the odds), it is easy to make designs with small R (such as $R = 8$ or $R = 16$).

It also makes sense, then, to design two experiments, say one to recover odd harmonics (eg $R = 8$ recovers harmonics $n = 1, 3, 5, 7, 9, \dots$ and another to recover most even harmonics (eg. $R = 9$ recovers harmonics $n = 1, 2, 4, 5, 7, 8, 10, 11, 13, \dots$ which include the even harmonics co-prime to 3. In this way, two experiments with a small number of tires, and a small number of controlled layers, can recover all the harmonics of interest.

We also pursued possible designs where the number of tires R is larger than the number of layers m which are being analysed. Once again, the $R \times R$ DFT matrix is the key idea, and so we select a subset of m columns from that DFT matrix for Z_1 to get an optimal design for the first harmonic. When n is co-prime to R, the nth harmonics matrix Z_n is also optimal, so again we can solve for a range of harmonics n which have no common divisors with R. (The factorization of m itself is irrelevant.) Is this better? Not really, as we have more tires to experiment with than number of layers to determine. But, it might be useful for the technicians designing the tests, as it tells them the range of angles they select in an experiment is determined by the number of tires, not by the number of layers.

Also, there is the possibility of selecting a subset of integers $h_1, h_2, \ldots, h_m$ from the full set $0, 1, \ldots, R-1$ as in the Simplified Lattice Method in such a way that the action of n on the differences $h_j - h_k \mod R$ is non-trivial. In such a case, the design matrix Z_n obtained from corresponding design angles will again be optimal. However, a few examples convinced us, that this was unlikely to lead to very efficient designs (i.e. number of tires R close to the number of layers m under control), and we abandoned this approach.

1.8.1 Extension by blocking

As noted above, there is a germ of an idea on how to cover harmonics of interest using a series of smaller test runs, rather than one large test. The advantages of this method include the simplification in angle setting for the test under consideration. That is, for a test with R tires, the angles to be set will be multiples of $360/R$ degrees, so for R small, we get a manageable set of test angles to work with. As an example, with $R = 8$, our test angles are multiples of 45 degrees, and the experiment will recover harmonics $1, 3, 5, 7, \ldots$. Repeating an experiment with $R = 9$ give angles as multiples of 40 degrees, and will recover harmonics $1, 2, 4, 5, 7, 8, 10, \ldots$. Thus with two experiments, using a total of 17 tires, we recover most the the first ten harmonics (we miss $n = 6$).

True, we could have run just one large test with 17 tires (17 is prime), but this would require test angle setting that are multiples of $360/17$ or 21.18 degrees. This is not a convenient number for real factory settings.

We explored this idea by considering a number of test cases: various layers, various numbers of tires, etc. Unfortunately, we misplaced our records of the cases we studied, but concluded we could often recover a wide range of harmonics by breaking up a large test (with unusual test angles) into two or three smaller tests, with better angles. The same, or slightly larger, number of test tires is all that is required in the sum of the tests. A tabulation of possibilities would be useful, which we could generate with some direction from Michelin researchers on what would be convenient tests.

1.9 Problem 2: Non-harmonic components

The second part of the problem described was to account for non-harmonic components in the measured waveform. We had some difficulty with this concept at first, as we understood the measurement to be cyclic (they represented forces around the circle of the tire) which by definition is periodic. Moreover, from the standard Fourier theory, any periodic waveform can be represented as a sum of harmonics, so there are no "non-harmonic" components. That is, we felt there would never be such a thing as a non-harmonic component, since the harmonic components were enough to describe everything.

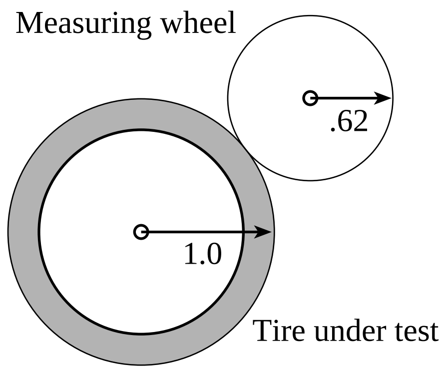

Figure 1.1: The force measuring wheel as it contacts the test wheel. Note the difference in dimension for the wheels.

However, this is a case of knowing too much mathematics for our own good! After a careful description of the construction and measurement methods used by Michelin for its tires, it became clear physically that such “non-harmonic” components could exist. For instance, as shown in Figure 1.1, the forces around the tire are measured by rolling a smaller wheel around the edge of the tire. Since the size of the measurement wheel might not be the same size as the tire under test, nor a simple ratio, the measurement process is not truly periodic! For example, the measurement wheel may not be perfectly round (neither is the test tire), and as the two roll against each other, the imperfections line up at different places, which may not be a periodic function of the test tire’s circumference.

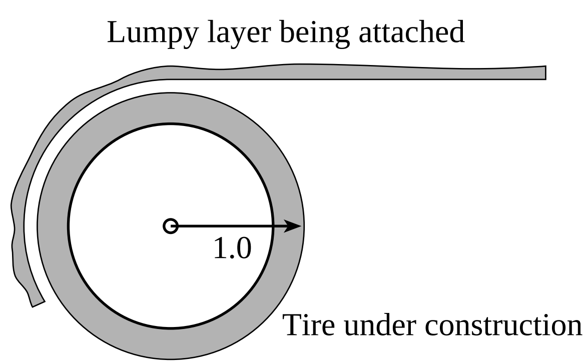

Figure 1.2: A lumpy, extruded layer being wrapped around the wheel.

Similarly, when the tire is being constructed, the component layers (rubber, cords, steel) may have some regular imperfections along their length – they may be “lumpy” for some reason, such as because of the extrusion process by which these layers are manufactured. The lumps, although regular, may not line up exactly along the circumference of the tire, and thus their forcing function is not really periodic. This is illustrated in Figure 1.2.

Although the harmonics are complete, we asked the question "what is the effect of analysing a non-harmonic component using only harmonics?" More precisely, we experimented with taking the usual discrete FFT of a signal that had one non-harmonic component, and observed its effects on the harmonic decomposition. For instance, in Figure 1.3, we see the first 10 Fourier components for a signal of the form $f(t) = \sin(2\pi t/.62)$ which can be thought of as a signal of period .62, or frequency 1.61 times the fundamental. Observe that, rather than a single spike (which we would expect of a signal with integer frequency multiple), we obtain a smear of energy in the first five or so low frequency components. For our tire problem, this says that a perfectly smooth tire, with a wobbly measurement wheel, would falsely exhibit large harmonics for $n = 1, 2, 3, 4, 5$. This is exactly the problem we wish to avoid.

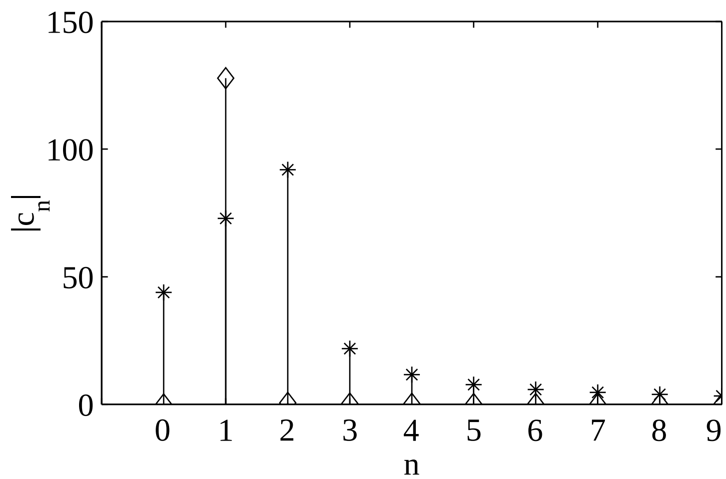

Figure 1.3: A stem plot of the modulus of the components of the Fourier spectrum for a non-harmonic signal with period 0.62. Rather than a single spike at frequency 1.61, we see it contaminates nearby frequency components $n = 0, 1, 2, 3, 4$. The diamonds correspond to the components of the waveform $\sin(2\pi t)$.

We also observed similar effects for the extrusion-type non-periodic variations, with contamination centred around frequencies close to the base frequency of the extrusion regularities.

This problem then has two parts: how do we detect and measure the strength of a non-harmonic component, and how do we design our experiments to account for these effects. Measuring the strength of the non-harmonic component is rather straight-forward. Once we know to expect such a component, we simply include it in our Fourier series decomposition of the force signature. For instance, in the case where the measurement wheel has a diameter .62 times the diameter of the tire under test, we expect a non-harmonic component at frequency 1/.62 times the fundamental. For $2N = 256$ samples the series expansion for $f(t)$ is taken to be

$$f(t_k) = \sum_{n=-127}^{128} c_n e^{2\pi i n t_k} + c_* e^{2\pi i t_k/.62} + c_{**} e^{-2\pi i t_k/.62}, \qquad t_k = \left\{ \frac{k}{256} \right\}_{k=0}^{255}.$$

The coefficients c_*, c_{**} represent the strength of the non-harmonic component. Of course, this is now an over-determined system of equations (256 equations in 258 unknowns), so to select a useful solution, we pick the one that minimizes the L^1 norm of the coefficients

$$\sum_{n=-127}^{128} |c_n| + |c_*| + |c_{**}|.$$

The choice of the L^1 norm forces the minimization routine to select a solution that is sparse; i.e. it concentrates the energy of the decomposition into as few components as possible. This is a well-known approach in frame theory.

We did some experiments in MATLAB with this approach and the relatively small dimensions ($d = 258$) gives a problem that is within the capabilities of the software. But, it is rather slow. Another approach is to use an undertermined system, by trying to express the force signature $f(t)$ as a sum of just a few harmonic and non-harmonic terms. For instance, using just the first 5 harmonic components, and one non-harmonic component, one can attempt to minimize

$$\left\| f(t_k) - \sum_{n=-5}^{5} c_n e^{2\pi i n t_k} - c_* e^{2\pi i \frac{t_k}{.62}} - c_{**} e^{-2\pi i \frac{t_k}{.62}} \right\|_2 + \alpha \left(\sum_{n=-5}^{5} |c_n| + |c_*| + |c_{**}| \right).$$

Here, we use the usual L^2 norm to measure the distance between the force f and its representation as a sum of harmonic and non-harmonic components and include a penalty term (scaled by α) which aims for sparseness of the representation.

When there are several non-harmonic components, we simply add on addition terms in the decomposition above. Note that this discussion assumes we know the frequency of the non-harmonic component. This would be the case of a measurement wheel of known dimension, or adding layers to the tire that have regular undulations (say due to extrusion) of known period. However, it is also possible to account for non-harmonic components of unspecfied frequency, by including terms like $e^{2\pi i \omega_n t}$ where ω_n is a parameter to be varied under the minimization step above. Of course, this is a much more challenging problem numerically.

Finally, we observed at the workshop that the usual Fourier decomposition of a signal can be generalized to a decomposition of any choice of basis functions with (non-harmonic) frequencies. We select some range of frequencies $\omega_n, n = 1, 2, \ldots, 256$ and solve the equation

$$f(t) = \sum_n c_n e^{2\pi i \omega_n t}.$$

For regularly spaced t, this is a Vandermonde matrix equation, for which there are fast linear solvers, and thus we need not shy away from inverting this 256×256 system of equations.

There remains the question of how to incorporate this non-harmonic component in the experimental design. For the problem with the measuring wheel, we are of the opinion that the non-harmonic component arising from measurement only serves to contaminate the true harmonics present in the tire under test. Thus, the approach should be to estimate the non-harmonic component using the techniques described here, subtract it from the signal, then perform the experiment as before. For the problem of non-harmonic components coming from extrusions, it seems these parts really do contribute to the performance of the tire – they are not simply a part of the measurement process. To design an experiment in this case, one must keep track not only of the position of the layer on the tire, but also the relative placement of the extruded irregularities. We did not have enough time to explore this in detail.

1.10 Problem 3: Combining waveforms

The third problem essentially asks: what happens when you measure several force waveforms? This would be the case, say, when several measuring wheels are used on one tire, perhaps one measuring radially, another perpendicular to the tread. Or, one might measure forces along three or four paths around the tread of the tire. The measured waveforms may be recorded separately, or they may be combined, say as a sum, or as a sum of squares.

After briefly considering this problem, we realized it had been essentially solved by the work above. The experimental design asks how to build test tires in order to determine the contribution of each component layer to some force profile $f(t)$. The analysis above generalizes from the case of a real-valued function (a single force), to the case of a vector-valued function (several forces, recorded simultaneously). There is no interaction between components, so we proceed with test tires as before. The analysis is then performed component-by-component. That is, we just look at each measured force as an independent result, and apply the methods discussed above.

In the case where the forces are summed together before analysis, we are back to the simple case of a single (real-valued) force function $f(t)$, and once again the earlier analysis works.

The complications occur when we take a sum of squares, for instance $f^2(t)+g^2(t)$. Of course, this results in a single real-valued function to analyse, but we have to observe that our Fourier decomposition in $f(t), g(t)$ will combine non-linearly. If we do a Fourier analysis of just the combined function, we expect some problems, since the underlying physical problem is now non-linear, and there should be interaction between the harmonics. In particular, this says our basic assumption of the linear model described in the earlier section will fail, and so we do not expect great success by this method.

On the other hand, we found it difficult to understand why one might combine force measurements in such a way. Since the analysis works for vector functions, we would recommend to Michelin to simple recover all measured force waveforms and work with them independently, rather than as some combined, lump sum result.

1.11 Problem 4: Extending to 2 dimensions

The original problem of 2000 looked at variations in the tire causes by translation of layers around the circumference of the tire. Problem 4 suggested we also consider variations caused by lateral displacement of the layers; that is, what happens when a layer is placed in different positions, varying in a direction perpendicular to the circumference of the tire. Michelin explained this could happen when laying down the steel belts - this layer could be position slightly to the left of centre-line, or slightly to the right. It is also possible to put down a layer at an angle, so the belt is to the left on part of the circumference, to the right on another part.

Several members of our groups spent quite a bit of time trying to extend our linear analysis described above to the two dimensional case. Namely, data would be recorded and a two-dimensional FFT would be attempted. At some point, we realized the 2D FFT model was really not appropriate, as the two dimensions are fundamentally different. For, around the tire, we have periodicity around the full circle. But, in the perpendicular direction, the layers can only be moved small amounts, and certainly never all the way around the cylinder of the tire. More precisely, we think of the tire as a two dimensional torus, or donut. We can move layers all the way around the large circumference of the torus, but only slightly in the cross-sectional direction.

A more appropriate model then would be to treat the circumferential direction via a 1D FFT, and model the variations in the transverse direction via a small order polynomial. That is, we expect small transverse movements in a layer to have a small effect on the harmonics, which could be modeled by a simple polynomial. Indeed, we could use a couple of small parameters to model two effects for each later: the distance from centre-line for the layer, and a "torque" amount, representing the amount of transverse twisting for the layer. Other effect due to small movements could be modeled similarly.

With this idea in mind, we see the experimental design utilizing FFT matrices in one dimension, and Vandermonde matrices (arising from the powers of x in the polynomial approximations) in the other dimension. We expect the two modes to be decoupled, so solving them would be straightforward.

However, we lacked sufficient time in the workshop to pursue all the details of this method. We believe the method is viable and worth pursuing.

1.12 Conclusions

Michelin presented us with four problems in the experimental design of a statistical test on manufactured tires. By extending the linear, Fourier model developed at the IPSW in 2000, we were able to make remarkable progress on the four problems, essentially resolving the first three, and suggesting a promising direction to resolve the fourth problem. It was a challenging and rewarding set of problems for our group to attack.

1.13 Acknowlegements

We would like to recognize the support from the Pacific Institute for the Mathematical Sciences, which organizes and funds these Industrial Problem Solving Workshops, and the University of British Columbia for hosting this year's workshop. We also would like to thank the people at the Michelin Corporation, and in particular Bill Mawby, for participating in the workshop, proposing the problems, and supporting our research.

Bibliography

[1] Aggarwala, R., Lamoureux, M. & Powojowski, M. (2002). *Statistical Design in Tire Manufacturing*. Canadian Applied Mathematics Quarterly, 10(2), pp. 179-198.

[2] Fang, K. T. & Wang, Y. (1994). *Number-theoretic Methods in Statistics*. Chapman & Hall: London.

[3] Körner, T. W. (1988). *Fourier Analysis*. Cambridge University Press: Cambridge.

[4] St. John, R. C. & Draper, N. R. (1975). *D-Optimality for Designs: A Review*. Technometrics, 17(1), pp. 15-22.

[5] Winker, P. & Fang, K. T. (1997). *Applications of Threshold-Accepting to the Evaluation of Discrepancy of a Set of Points*. SIAM Journal of Numerical. Analysis, 34(5), pp. 2028-2042.