Multi-Name Credit Derivatives

Pat Hagan
Nomura Securities International, Inc.
Industry Representative

Kristen Campbell, Rensselær Polytechnic Institute
Yun Chen, University of Delaware
David A. Edwards, University of Delaware
Yanyan Li, University of Delaware
Sean O'Connell, Rensselær Polytechnic Institute
Ryshon Patterson, Rensselær Polytechnic Institute
Gilberto Schleiniger, University of Delaware
Jodi Schneider, University of Texas at Austin
Fabricio Tourrucoo, University of Delaware
Gehua Yang, Rensselær Polytechnic Institute
Juan-Ming Yuan, University of Texas at Austin
and others...

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Section 1: Introduction

The problem addressed in this report is that of pricing multi-name credit derivatives. These are default guarantee contracts on a basket of "names" whose default rates are correlated.

To understand the usefulness of such derivatives and the main difficulties in pricing them, we briefly describe their main characteristics and reasonable assumptions that can be made in formulating a pricing method for multi-name credit derivatives.

The type of contract that we wish to address is the following: An investor desires to invest in bonds of a basket of *names*, but wants to be protected in the event of default. One examples of default protection is the "first-to-default" guarantee, which pays only the loss associated with the first name in the basket to default. In the case of *traunche guarantee protection* for CDO (consolidated debt options) or CLO (consolidated loan options), the protection pays off only if enough of the underlying commercial mortgages default to hurt the guarantee traunche of the bond.

The main difficulty in pricing and hedging multi-name deals is the lack of good joint credit models that account for the correlations in the credit worthiness of the names.

Let $f(t,T)$ be the forward rate for T as seen at t, and $r(t) = f(t,t)$ be the short rate. Let

$$Z(t,T) = e^{-\int_t^T f(t,\tau)\,d\tau} = \text{value at } t \text{ of \$1 paid at } T.$$

The discount factor is

$$D(T) = Z(0,T) = \text{today's value of \$1 paid at } T.$$

By the fundamental theorem of no-arbitrage finance, there is a probability measure such that the value $V(t)$ of any deal is

$$V(t) = E\left\{ e^{-\int_t^T r(\tau)\,d\tau} V(T) \;\middle|\; t \right\}$$

if there are no cash flows. In other words, the value at time t is the value at time T discounted for the period $t < \tau < T$. If there are cash flows $C(t)$ that are paid out over time, then those must be discounted as well, and we have

$$V(t) = E\left\{ e^{-\int_t^T r(\tau)\,d\tau} V(T) + \int_t^T e^{-\int_t^{T'} r(\tau)\,d\tau} C(T')\,dT' \;\middle|\; t \right\}.$$

Let

$$Q_j(t,T) = \text{probability that company } j \text{ has not defaulted at } T \text{ as seen at } t.$$

Then

$$Q_j^0(T) = Q_j(0,T) = \text{today's probability that } j \text{ as not defaulted at } T.$$

It is assumed that all default probabilities are independent of interest rates. Then the no default risk value of \$1 to be paid at T is

$$Z(t,T) = E\{\, e^{-\int_t^T r(\tau)\, d\tau}\, \$1 \mid t \,\}.$$

Consider a *bullet bond* on company j. In such a bond, a *coupon* (interest) amount c_i is paid at t_i , $i = 1 \ldots n-1$, if the company does not default on or before t_i , and $1 + c_n$ (the last coupon plus the original \$1 investment) is paid on t_n if the company does not default on or before t_n. In case of default, the bond has a recovery value of w paid on t_i if default occurs in $t_{i-1} < t \le t_i$. The value of the bond today (to the holder) is

$$V_{\text{bond}}(0) = \sum_{i=1}^{n} c_i D(t_i) Q_j^0(t_i) + D(t_n) Q_j^0(t_n) + w \sum_{i=1}^{n} D(t_i)[Q_j^0(t_{i-1}) - Q_j^0(t_i)]$$

Examples of credit derivatives on a single name are:

- Default guarantee on company j for $t_0 < t < t_m$. This stipulates that a fee a_i is paid at t_i , $i = 1 \ldots m$ if the company does not default on or before t_i, and $1 - w$ is paid to the client if company j defaults in $t_{i-1} < t \le t_i$. Today's value of this guarantee (to the guarantor) is

$$V_{\text{guar}}(0) = \sum_{i=1}^{m} a_i D(t_i) Q_j^0(t_i) - (1-w) \sum_{i=1}^{m} D(t_i)[Q_j^0(t_{i-1}) - Q_j^0(t_i)].$$

- Digital guarantee. This is a special case of the default guarantee. Here the guarantor does not retain any portion w of the bond amount on default, and instead the bond-holder is refunded the full value of the bond. Thus we take $w = 0$ in the above to obtain

$$V_{\text{guar}}(0) = \sum_{i=1}^{m} a_i D(t_i) Q_j^0(t_i) - \sum_{i=1}^{m} D(t_i)[Q_j^0(t_{i-1}) - Q_j^0(t_i)].$$

Examples of multi-name derivatives are:

- Basket protection for $t_0 < t < t_m$ written on names $j = 1, 2, \ldots J$. Under this contract, a fee a_i is paid at t_i , $i = 1 \ldots m$ if no company defaults on or before t_i, and $1 - w_j$ is paid to the client if company j defaults in $t_{i-1} < t \le t_i$ and j is the first to default.
- Traunche protection for CDO and CLO. This is similar to the basket protection described above, except that the fees a_i continue to be paid until enough companies in the basket default to trigger the payout.

Section 2: Credit Derivatives

We begin by deriving the price for a *default guarantee contract*, more specifically a guarantee on a single company. Suppose that a client wishes to invest in the bonds of a company, but wants some amount of insurance against issuer default. Such an investor may purchase a default guarantee from an investment firm.

Let t_i, $i = 0, 1, \ldots, M$ be the dates on which interest payments are supposed to be paid by the issuer, and also let us normalize to consider a bond with principal amount 1. (All quantities scale linearly with the principal.) The investor pays the investment firm a premium ϕ_i on t_i when the issuer pays the interest. (We assume that the interest due and premium due dates are the same for simplicity, though in practice if the investment firm is handling the bond purchase, the premium may simply be deducted from the interest payments.)

However, if the issuer defaults in the interval $(t_{i-1}, t_i]$, then the issuer won't make the interest payment on t_i, and the investment firm must pay the investor an amount $1 - w$. Here w is some amount (usually taken to be 0.2 to 0.25) that the investor should be able to recoup from the company during a bankruptcy proceeding, and hence the two payments together sum to the original principal amount 1. In summary, from the investment firm's point of view, it will

- receive ϕ_i at time t_i if the company (numbered 0) doesn't default by t_i
- pay $1 - w$ at time t_i if company 0 defaults in $(t_{i-1}, t_i]$.

Let τ_0 be the time at which company 0 defaults. Then we define the *survival probability distribution function* $Q_0(t)$ for company 0 in the following way:

$$\begin{aligned} Q_0(t) &= \text{probability that company 0 hasn't defaulted by time } t \\ &= P(t < \tau_0). \end{aligned} \tag{2.1}$$

We note that given this definition, the probability that the company defaults in $(t_{i-1}, t_i]$ is given by

$$P(t_{i-1} < \tau_0 \leq t_i) = P(t_{i-1} < \tau_0) - P(t_i < \tau_0) = Q_0(t_{i-1}) - Q_0(t_i). \tag{2.2}$$

Thus, the value V_0 of the guarantee today (always taken to be $t = 0$) is given by

$$\begin{aligned} V_0 = &\sum_{i=1}^{M} (\text{present value of } \phi_i \text{ paid at time } t_i) Q_0(t_i) \\ &- \sum_{i=1}^{M} (\text{present value of } 1 - w \text{ paid at time } t_i)[Q_0(t_{i-1}) - Q_0(t_i)]. \end{aligned} \tag{2.3}$$

To calculate the present value of an amount of money payable at time t, we multiply by the *discount factor* $D(t)$. Thus, (2.3) becomes

$$\begin{aligned} V_0 &= \sum_{i=1}^{M} \phi_i D(t_i) Q_0(t_i) - \sum_{i=1}^{M} (1-w) D(t_i) [Q_0(t_{i-1}) - Q_0(t_i)] \\ &= \sum_{i=1}^{M} D(t_i) \left[(1+\phi_i - w) Q_0(t_i) - (1-w) Q_0(t_{i-1}) \right]. \end{aligned} \tag{2.4}$$

Since the discount factor is assumed known, the problem boils down to determining $Q_0(t)$ for the company.

If investors wish to insure the bonds for n companies, there are several types of protection they can buy. We wish to focus on *first-to-default* protection. Here the premium is paid if *all* the companies make their interest payments. Otherwise, if company j defaults, $1-w_j$ is paid by the investment firm, and the policy terminates. Note that if two companies default in the same time frame, only one payout is made. In practice, the payout is made immediately upon default.

Thus the bulleted text above is replaced by

- receive ϕ_i at time t_i if *no* company has defaulted by t_i
- pay $1 - w_j$ at time t_i if first default (by company j) is in $(t_{i-1}, t_i]$.

We now must redefine some terms to obtain the form analogous to (2.4). Let τ_j be the time at which the jth company defaults. Then we define $Q(t)$, the *joint survival probability distribution function*, in an analogous way to the above as follows:

$$\begin{aligned} Q(t) &= \text{probability no company has defaulted at time } t \\ &= P(t < \tau_j \ \forall j). \end{aligned} \tag{2.5}$$

Next we define the probability that company j has not defaulted in the interval $[0, t]$, but that when it does default, it will be the first:

$$Q_j^{\mathrm{f}}(t) = P(t < \tau_j \text{ and } \tau_k > \tau_j \ \forall k \neq j), \tag{2.6}$$

where the superscript 'f' refers to 'first.' We note that given this definition, the probability that company j has defaulted in the interval $(t_{i-1}, t_i]$, and that it is the first to default is given by

$$\begin{aligned} P(t_{i-1} < \tau_j \leq t_i \text{ and } \tau_k > \tau_j \ \forall k \neq j) &= P(t_{i-1} < \tau_j \text{ and } \tau_k > \tau_j \ \forall k \neq j) \\ &\quad - P(t_i < \tau_j \text{ and } \tau_k > \tau_j \ \forall k \neq j) \\ &= Q_j^{\mathrm{f}}(t_{i-1}) - Q_j^{\mathrm{f}}(t_i). \end{aligned} \tag{2.7}$$

When valuing the contract, we must now consider that any of the n different companies might default first, and depending on which does, we have a different payout function.

Thus (2.4) becomes

$$V = \sum_{i=1}^{M} \phi_i D(t_i) Q(t_i) - \sum_{i=1}^{M} D(t_i) \sum_{j=1}^{n} (1 - w_j)[Q_j^{\mathrm{f}}(t_{i-1}) - Q_j^{\mathrm{f}}(t_i)]$$
$$= \sum_{i=1}^{M} D(t_i) \left\{ \phi_i Q(t_i) - \sum_{j=1}^{n} (1 - w_j)[Q_j^{\mathrm{f}}(t_{i-1}) - Q_j^{\mathrm{f}}(t_i)] \right\}. \tag{2.8}$$

Equation (2.8) simplifies somewhat in the case where $w_j = w \ \forall j$:

$$V = \sum_{i=1}^{M} D(t_i) \left\{ \phi_i Q(t_i) - (1 - w) \sum_{j=1}^{n} Q_j^{\mathrm{f}}(t_{i-1}) - Q_j^{\mathrm{f}}(t_i) \right\}. \tag{2.9}$$

However, we note that

$$Q(t) = P(t < \tau_j \ \forall j) = \sum_{j=1}^{n} P(t < \tau_j \text{ and } \tau_k > \tau_j \ \forall k \neq j) = \sum_{j=1}^{n} Q_j^{\mathrm{f}}(t),$$

where we assume that the probability of two companies defaulting at the same time is zero. Substituting this result into (2.9), we have

$$V = \sum_{i=1}^{M} D(t_i) \left\{ \phi_i Q(t_i) - (1 - w)[Q(t_{i-1}) - Q(t_i)] \right\},$$

which is exactly in the form of (2.4), though now Q refers to a probability under a *joint* survival function. The remainder of this report deals with constructing appropriate forms for Q given forms for the Q_j,

$$Q_j(t) = P(t < \tau_j).$$

This is the probability of company j defaulting if none of the other companies were considered in the analysis (the marginal survival probability distribution).

Section 3: (Too) Simple Models

The discount factor is usually derived by introducing the *instantaneous forward rate* $f(t)$. If you invest \$1 at time t, then you will receive $e^{f(t)\,dt}$ at time $t+dt$. Then continually re-investing the money throughout the time period $[0,t]$, one has that at the end of the period the

$$\text{future value at } t \text{ of \$1 invested now} = \exp\left(\int_0^t f(\xi)\,d\xi\right).$$

(Here f is a deterministic function because the rate is locked in once the contract is signed.) Thus by simply saying that we know the future value, we have that

$$(\text{present value of \$1 payable at time } t) = D(t) = \exp\left(-\int_0^t f(\xi)\,d\xi\right). \tag{3.1}$$

In order to use the survival probability results easily on computer systems designed to handle discount calculations, one can define the *default rate* $h(t)$ in the following way:

$$Q(t) = \exp\left(-\int_0^t h(\xi)\,d\xi\right). \tag{3.2}$$

This then relates the probability $Q(t)$ to a *Poisson process*, which models waiting times, time to failure (which of course is directly analogous to our case), etc. Similarly, we can define a default rate for company j through the relationship

$$Q_j(t) = \exp\left(-\int_0^t h_j(\xi)\,d\xi\right). \tag{3.3}$$

If the default rates of the companies are independent of each other, then we know that

$$\begin{aligned}
P(t < \tau_j\ \forall j) &= \prod_{j=1}^{n} P(t < \tau_j) \\
Q(t) &= \prod_{j=1}^{n} Q_j(t) \\
\exp\left(-\int_0^t h(\xi)\,d\xi\right) &= \prod_{j=1}^{n} \exp\left(-\int_0^t h_j(\xi)\,d\xi\right) \\
h(t) &= \sum_{j=1}^{n} h_j(t).
\end{aligned} \tag{3.4}$$

We call (3.4) the *independent default approximation.*

We begin by considering the case where h_j is a constant, and we have only two companies. Then if the default rates are independent, we have

$$Q_j(t) = e^{-h_j t}, \quad j = 1, 2, \tag{3.5a}$$

$$Q(t) = e^{-(h_1+h_2)t}. \tag{3.5b}$$

Motivated by the simple form of (3.5b), we tried to introduce expressions for $Q(t)$ similar in form to (3.5b) but which included the relationship between the defaults of companies 1 and 2. To examine the effect, we let X_j be the following *indicator variable*:

$$X_j = \begin{cases} 1, & \text{if company } j \text{ has not defaulted (probability } Q_j(t)) \\ 0, & \text{else.} \end{cases} \tag{3.6}$$

Then to quantify the relationship between defaults, we define the *conditional* probability $Q_j^{\mathrm{c}}(t)$ by

$$Q_j^{\mathrm{c}}(t) = P(\text{company j hasn't defaulted given that all others haven't}).$$

Therefore, for the two-company case, we have

$$Q_1^{\mathrm{c}}(t) = P(X_1 = 1 | X_2 = 1) = \frac{P(X_1 = 1, X_2 = 1)}{P(X_2 = 1)} = \frac{Q(t)}{Q_2(t)}, \tag{3.7a}$$

$$Q_2^{\mathrm{c}}(t) = \frac{Q(t)}{Q_1(t)}. \tag{3.7b}$$

From (3.5) clearly we have that in the independent case, $Q_1^{\mathrm{c}}(t) = Q_1(t)$ and $Q_2^{\mathrm{c}}(t) = Q_2(t)$.

Given any two variables X_1 and X_2, their *correlation coefficient* $\rho(X_1, X_2)$ is given by

$$\mathrm{E}(X_1 X_2) = \mathrm{E}(X_1)\mathrm{E}(X_2) + \rho(X_1, X_2)\sqrt{\sigma_1^2 \sigma_2^2}, \tag{3.8}$$

where σ_j is the standard deviation of X_j. (In the financial parlance, σ_j is called the *volatility.*) Using basic facts about indicator variables, we have

$$\begin{aligned} P(X_1 = 1 \text{ and } X_2 = 1) &= P(X_1 = 1)P(X_2 = 1) \\ &\quad + \rho(X_1, X_2)\sqrt{P(X_1 = 1)[1 - P(X_1 = 1)]P(X_2 = 1)[1 - P(X_2 = 1)]}. \end{aligned}$$

Then using the known relationships between the probabilities and the Qs, we have

$$\begin{aligned} Q(t) &= Q_1(t)Q_2(t) + \rho(X_1, X_2)\sqrt{Q_1(t)Q_2(t)[1 - Q_1(t)][1 - Q_2(t)]} \\ \rho &= \frac{Q(t) - Q_1(t)Q_2(t)}{\sqrt{Q_1(t)Q_2(t)[1 - Q_1(t)][1 - Q_2(t)]}}. \end{aligned}$$

In each of our attempts at modeling a form for $Q(t)$, we want to keep the Poisson form for the companies treated separately. Thus we retain (3.5a), which yields

$$\rho(X_1, X_2) = \frac{Q(t) - e^{-(h_1+h_2)t}}{e^{-(h_1+h_2)t/2}\sqrt{(1-e^{-h_1 t})(1-e^{-h_2 t})}}. \tag{3.9}$$

Since $\rho(X_1, X_2)$ is always a complicated function of time, we focused more on the effect of various guesses for $Q(t)$ on the $Q_j^{\mathrm{c}}(t)$.

One postulate for such a function is also

$$Q(t) = e^{-ht}. \tag{3.10}$$

With such a postulate, we have that

$$Q_1^{\mathrm{c}}(t) = \frac{e^{-ht}}{e^{-h_2 t}} = e^{-h_1 t}e^{-(h-h_1-h_2)t} = e^{-(h-h_1-h_2)t}\, Q_1(t), \tag{3.11a}$$

$$Q_2^{\mathrm{c}}(t) = e^{-(h-h_1-h_2)t}\, Q_2(t). \tag{3.11b}$$

Since Q_1^{c} must be a legitimate distribution function, the argument of the exponential must be negative so we have $Q_1^{\mathrm{c}}(\infty) = 0$. Then using our expression in (3.11a), we have

$$h > h_2.$$

But Q_2^{c} must be a legitimate distribution function as well, so the same corresponding bound must be satisfied. Thus we have

$$h > \max\{h_1, h_2\} \geq \frac{h_1 + h_2}{2}. \tag{3.12}$$

Unfortunately, there are problems with this approach. First, we could find no way to justify, either on stochastic or financial grounds, the form (3.10). In adddition, we found that the correlation coefficient behaves in an unsatisfactory manner. For instance, substituting (3.10) into (3.9), we obtain

$$\rho(X_1, X_2) = \frac{e^{-ht} - e^{-(h_1+h_2)t}}{e^{-(h_1+h_2)t/2}\sqrt{(1-e^{-h_1 t})(1-e^{-h_2 t})}} = \frac{e^{-(h-(h_1+h_2)/2)t} - e^{-(h_1+h_2)t/2}}{\sqrt{(1-e^{-h_1 t})(1-e^{-h_2 t})}}$$

$$\lim_{t\to\infty} \rho(X_1, X_2) = 0,$$

in view of (3.12). However, this would seem to be incompatible with financial reality, since if two companies are in the same sector and hence correlated, their correlation should not decrease over time if they remain in the same core businesses.

Section 4: The Feynman-Kac Formula

In order to motivate our choice of joint survival distribution function, we let $H_j(t)$ be a random variable representing the default rate of company j at time t. How then do we relate the random quantity H_j to the deterministic quantities of interest Q_j and h_j? The answer lies in the *Feynman-Kac Formula*. We could simply quote the result, but the introduction of correlation complicates matters somewhat, so we derive selected portions here.

Let the random variable H_j follow the diffusion process

$$dH_j = g_{j,1}(\mathbf{H})\,dt + g_{j,2}(\mathbf{H})\,dW_j, \tag{4.1a}$$

where dW_j is a Wiener process and the gs are arbitrary functions of $\mathbf{H}$, the vector of the H_j. We correlate the H_j by letting

$$dW_j\,dW_k = \rho_{jk}\,dt, \tag{4.1b}$$

where ρ_{jk} is the correlation between H_j and H_k. Of course, $\rho_{jj} = 1$ because every variable is correlated with itself and $\rho_{jk} = \rho_{kj}$ because the order of the Ws doesn't matter.

Now for any function $G(\mathbf{H}, t)$, we wish to calculate dG. We present some heuristic arguments; more details may be found in [1]. We note from (4.1b) that dW_j is roughly $O(\sqrt{dt})$. Therefore, we have that

$$\begin{aligned}
dG &= \frac{\partial G}{\partial t}\,dt + \sum_{k=1}^{n}\frac{\partial G}{\partial H_k}\,dH_k + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 G}{\partial H_k H_l}\,dH_k\,dH_l + o(dt) \\
dG &= \frac{\partial G}{\partial t}\,dt + \sum_{k=1}^{n}\frac{\partial G}{\partial H_k}\,dH_k + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 G}{\partial H_k H_l}(g_{k,2}(\mathbf{H}))(g_{l,2}(\mathbf{H}))\,dW_k\,dW_l + o(dt) \\
&= \frac{\partial G}{\partial t}\,dt + \left(\sum_{k=1}^{n} g_{k,1}(\mathbf{H})\frac{\partial G}{\partial H_k} + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\rho_{kl}(g_{k,2}(\mathbf{H}))(g_{l,2}(\mathbf{H}))\frac{\partial^2 G}{\partial H_k H_l}\right)dt \\
&\quad + \sum_{k=1}^{n} g_{k,2}(\mathbf{H})\frac{\partial G}{\partial H_k}\,dW_k + o(dt).
\end{aligned} \tag{4.2}$$

The term in the parentheses is called the *generator* of the diffusion.

By the Feynman-Kac Lemma [2], if G has the following form:

$$G(\mathbf{x}, t) = \mathrm{E}\left\{\exp\left(-\int_0^t g_3(\mathbf{H}(\xi))\,d\xi\right)\middle|\,\mathbf{H}(0) = \mathbf{x}\right\}, \tag{4.3}$$

then the value of G is related to the generator in the following way:

$$-\frac{\partial G}{\partial t}+\sum_{k=1}^{n} g_{k,1}(\mathbf{x})\frac{\partial G}{\partial x_k}+\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\rho_{kl}(g_{k,2}(\mathbf{x}))(g_{l,2}(\mathbf{x}))\frac{\partial^2 G}{\partial x_k \partial x_l}=g_3(\mathbf{x})G, \tag{4.4a}$$

where the extra minus sign in the t derivative comes from the minus sign in the exponential in (4.4). (In particular, in financial papers it is customary to denote the beginning time as t and the final time as T, which then removes the minus sign from both places.) Since the function G is known at $t=0$, the initial condition for the problem becomes

$$G(\mathbf{x},0)=1. \tag{4.4b}$$

Given the definitions in (4.1a), $Q_j(t)$ can be interpreted as

$$\begin{aligned} Q_j(t) &= \text{expected survival probability distribution function at time } t \\ &\qquad \text{over all possible paths } H_j(t) \text{ given } \mathbf{H} \text{ now} \\ Q_j(\mathbf{x},t) &= \mathrm{E}\left\{\mathcal{Q}_j(H_j)\middle|\,\mathbf{H}(0)=\mathbf{x}\right\}, \end{aligned}$$

where $\mathcal{Q}_j$ is a random variable corresponding to Q_j, namely

$$\mathcal{Q}_j(H_j)=\exp\left(-\int_0^t H_j(\xi)\,d\xi\right).$$

Note that Q_j is a deterministic result, since it is an expectation value. But then using the definition of the default rate, we have that

$$Q_j(\mathbf{x},t)=\mathrm{E}\left\{\exp\left(-\int_0^t H_j(\xi)\,d\xi\right)\middle|\,\mathbf{H}(0)=\mathbf{x}\right\},$$

and thus we may apply the Feynman-Kac Lemma with $G=Q_j$, $g_3(\mathbf{x})=x_j$ to obtain

$$-\frac{\partial Q_j}{\partial t}+\sum_{k=1}^{n} g_{k,1}(\mathbf{x})\frac{\partial Q_j}{\partial x_k}+\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\rho_{kl}(g_{k,2}(\mathbf{x}))(g_{l,2}(\mathbf{x}))\frac{\partial^2 Q_j}{\partial x_k \partial x_l}=x_j Q_j, \tag{4.5a}$$

$$Q_j(\mathbf{x},0)=1. \tag{4.5b}$$

Similarly, $Q(t)$ can be interpreted as

$$\begin{aligned} Q(t) &= \text{expected survival probability at time } t \text{ over all possible paths } \mathbf{H}(t) \text{ given } \mathbf{H} \text{ now} \\ Q(\mathbf{x},t) &= \mathrm{E}\left\{\mathcal{Q}(\mathbf{H})\middle|\,\mathbf{H}(0)=\mathbf{x}\right\}. \end{aligned}$$

But how do we relate $\mathcal{Q}$ to the $\mathcal{Q}_j$? Recall that the companies are independent entities; thus the *event* of failure is independent for each company. (It is just the random variables

corresponding to the failure rates that are correlated.) Then we see that the probability of the entire basket failing *exactly* at time t is given by

$$P(\tau_j = t \text{ for every } j) = \prod_{j=1}^{n} P(\tau_j = t) = \prod_{j=1}^{n} \frac{d[1 - Q_j(\tau_j)]}{dt},$$

where the first equality holds because of the independence and the second because Q_j is a distribution function. Continuing to simplify, we have

$$P(\tau_j = t \text{ for every } j) = \prod_{j=1}^{n} H_j \exp\left(- \int_0^t H_j(\xi)\, d\xi \right),$$

Again using the fact that the event of failure is independent for each company, we may integrate the above to obtain

$$\begin{aligned} P(t < \tau_j \ \forall j) &= \int_t^\infty \prod_{j=1}^{n} H_j \exp\left(- \int_0^{t'} H_j(\xi)\, d\xi \right) dt' \\ &= \prod_{j=1}^{n} \int_t^\infty H_j \exp\left(- \int_0^{t'} H_j(\xi)\, d\xi \right) dt' = \prod_{j=1}^{n} \exp\left(- \int_0^t H_j(\xi)\, d\xi \right) \\ &= \exp\left(- \int_0^t \sum_{j=1}^{n} H_j(\xi)\, d\xi \right). \end{aligned} \tag{4.6}$$

But the probability listed in (4.6) is just $\mathcal{Q}(t)$. Thus the random variables $\mathcal{Q}$ and $\mathcal{Q}_j$ behave as if they are independent; they are coupled only through the diffusion process in (4.1a), which expresses itself in the generator in the Feynman-Kac Lemma.

Thus we may apply the Feynman-Kac Lemma with $G = Q$, $g_3(\mathbf{x}) = \sum x_j$ to obtain

$$-\frac{\partial Q}{\partial t} + \sum_{k=1}^{n} g_{k,1}(\mathbf{x}) \frac{\partial Q}{\partial x_k} + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \rho_{kl}(g_{k,2}(\mathbf{x}))(g_{l,2}(\mathbf{x})) \frac{\partial^2 Q}{\partial x_k \partial x_l} = Q \sum_{j=1}^{n} x_j, \tag{4.7a}$$

$$Q(\mathbf{x}, 0) = 1. \tag{4.7b}$$

Section 5: Straight Wiener Process

We begin with the simplest case, letting H_j follow a random walk:

$$dH_j = \sigma_j \, dW_j, \tag{5.1a}$$

so we have

$$g_{j,1}(\mathbf{H}) = 0, \qquad g_{j,2}(\mathbf{H}) = \sigma_j. \tag{5.1b}$$

Substituting (5.1b) into (4.5a), we obtain

$$-\frac{\partial Q_j}{\partial t} + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \rho_{kl}\sigma_k\sigma_l \frac{\partial^2 Q_j}{\partial x_k \partial x_l} = x_j Q_j. \tag{5.2}$$

Since the initial condition (4.5b) is independent of $\mathbf{x}$ and the right-hand side has only an x_j term in it, we try a solution of the following form:

$$Q_j(\mathbf{x}, t) = Q_j(x_j, t). \tag{5.3}$$

Substituting (5.3) into (5.2) and (4.5b), we have

$$-\frac{\partial Q_j}{\partial t} + \frac{\sigma_j^2}{2}\frac{\partial^2 Q_j}{\partial x_j^2} = x_j Q_j, \tag{5.4a}$$

$$Q_j(x_j, 0) = 1. \tag{5.4b}$$

One of the main goals is to construct the h_j. Therefore, motivated by the exponential form in (3.3), we let

$$Q_j(x_j, t) = \exp\left(-b_{jj}(t)x_j - b_{0j}(t)\right), \qquad b_{0j}(0) = b_{jj}(0) = 0, \tag{5.5}$$

with the bs ≥ 0 and where the initial conditions have been chosen to satisfy (5.4b). (The reason for the double subscript will become clear in section 7.) We note from (3.3) that with this *ansatz*, the default rate for company j becomes

$$h_j = b'_{jj}x_j + b'_{0j}. \tag{5.6}$$

Substituting (5.5) into (5.4a) and using (5.6), we obtain

$$-\left(-x_j b'_{jj} - b'_{0j}\right) Q_j + \frac{\sigma_j^2}{2} b_{jj}^2 Q_j = x_j Q_j \tag{5.7a}$$

$$h_j = x_j - \frac{\sigma_j^2}{2} b_{jj}^2. \tag{5.7b}$$

Note from (5.7b) that we never have to solve for b_{0j}! We use (5.7b) later to simplify our algebra; to obtain a solution, we match coefficients of x_j in (5.7a) to obtain

$$\begin{aligned} b'_{jj} &= 1 \\ b_{jj} &= t, \end{aligned} \tag{5.8}$$

$$h_j(t) = x_j - \frac{\sigma_j^2 t^2}{2} \tag{5.9}$$

$$Q_j(x_j, t) = \exp\left(-x_j t + \frac{\sigma_j^2 t^3}{6}\right).$$

where we have used (5.7b) and (3.3). But this probability blows up as $t \to \infty$! The key to resolving the discrepancy lies in the default rate.

Equation (5.9) allows the default rate to go negative, which we know cannot be the case. This happens because (5.1a) models a standard Wiener process. This means there is no limit on the value of H_j as Brownian motion ensues. Since the variance increases with t, it is more and more likely that h_j will go negative for some values of t. Clearly a negative default rate (corresponding to businesses which spontaneously emerge from default) is not realistic. In such a case, we see by (3.3) that the survival probability distribution function would *increase*, another unrealistic result (leading to a probability> 1 — nonsense).

However, we may interpret (5.9) as the Taylor expansion of a function $h_j(t)$ in the limit of small time (or equivalently, low volatilities). It starts out with quadratic terms because there is no drift. However, for the Taylor series approximation to make sense, we should use this expression only for

$$\frac{\sigma_j^2 t^2}{2} \ll x_j.$$

However, if we restrict time to satisfy the above, it is unclear whether the result would be useful in the financial context.

Now that we have the Q_j and h_j, we want to calculate Q and h to see how the correlation affects the default independence result. Substituting (5.1b) into (4.7a), we obtain

$$-\frac{\partial Q}{\partial t} + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\rho_{kl}\sigma_k\sigma_l\frac{\partial^2 Q}{\partial x_k \partial x_l} = Q\sum_{j=1}^{n} x_j, \tag{5.10}$$

If we define the *covariance matrix* C by

$$c_{kl} = \rho_{kl}\sigma_k\sigma_l, \tag{5.11}$$

then

$$\nabla\cdot(C\nabla Q) = \sum_{k=1}^{n}\frac{\partial}{\partial x_k}\sum_{l=1}^{n} c_{kl}\frac{\partial Q}{\partial x_l} = \sum_{k=1}^{n}\sum_{l=1}^{n}\rho_{kl}\sigma_k\sigma_l\frac{\partial^2 Q}{\partial x_k\partial x_l}. \tag{5.12}$$

Thus (5.10) may be rewritten as

$$-\frac{\partial Q}{\partial t} + \frac{1}{2}\nabla\cdot(C\nabla Q) = Q\sum_{j=1}^{n} x_j. \tag{5.13}$$

Since we expect Q to be very close to the value given if the probabilities were independent (this is a weak correlation assumption), we let

$$Q = \exp\left(-b_\rho(t)\right) \prod_{j=1}^{n} Q_j, \qquad b_\rho(0) = 0, \tag{5.14}$$

where the initial condition has been chosen to satisfy (4.7b). Here we use the subscript ρ since this is the term that is going to include the effects of the correlations. Clearly if $b_\rho = 0$, then the defaults are independent. We also note from this definition that

$$Q(\mathbf{x}, t) = \exp\left(-\int_0^t h(\xi)\, d\xi\right) = \exp\left(-\int_0^t \left(b'_\rho(\xi) + \sum_{j=1}^{n} h_j(\xi)\right) d\xi\right) \tag{5.15a}$$

$$h(t) = b'_\rho(t) + \sum_{j=1}^{n} h_j(t). \tag{5.15b}$$

If $b_\rho = 0$, then $b'_\rho = 0$ and hence the default rates are independent as well.

As preparatory steps, we note from (5.15a) that

$$\frac{\partial Q}{\partial t} = -\left(b'_\rho + \sum_{j=1}^{n} h_j\right) Q, \tag{5.16a}$$

$$= -\left[b'_\rho + \sum_{j=1}^{n} \left(x_j - \frac{\sigma_j^2}{2} b_{jj}^2\right)\right] Q, \tag{5.16b}$$

$$\frac{\partial Q}{\partial x_k} = \left(\frac{\partial Q_k}{\partial x_k}\right) \frac{Q}{Q_k} = -b_{kk} Q, \tag{5.17a}$$

$$\frac{\partial^2 Q}{\partial x_k \partial x_l} = -b_{kk} \frac{\partial Q}{\partial x_l} = b_{kk} b_{ll} Q, \tag{5.17b}$$

where in deriving (5.16b) we have used (5.7b), and in deriving (5.17) we have used (5.5). Substituting (5.15a) into (5.10) and using (5.16b) and (5.17), we have

$$\left[b'_\rho + \sum_{j=1}^{n} \left(x_j - \frac{\sigma_j^2}{2} b_{jj}^2\right)\right] Q + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \rho_{kl} b_{kk} b_{ll} \sigma_k \sigma_l Q = Q \sum_{j=1}^{n} x_j$$

$$b'_\rho + \frac{1}{2} \sum_{k=1}^{n} \sum_{l \neq k} \rho_{kl} \sigma_k \sigma_l b_{kk} b_{ll} = 0$$

$$b'_\rho = -\frac{1}{2} \sum_{k=1}^{n} \sum_{l \neq k} \rho_{kl} \sigma_k \sigma_l b_{kk} b_{ll}, \tag{5.18}$$

where in performing the cancellation we have used the fact that j, k, and l are dummy variables. Now if we let

$$\mathbf{b}_k = b_{kk}\mathbf{e}_k, \tag{5.19}$$

then

$$\mathbf{b}_k^T C \mathbf{b}_l = b_{kk} b_{ll} \mathbf{e}_k^T C \mathbf{e}_l = b_{kk} b_{ll} c_{kl} = b_{kk} b_{ll} \rho_{kl} \sigma_k \sigma_l. \tag{5.20}$$

Upon substitution of the above into (5.18), we have

$$b'_\rho = -\frac{1}{2} \sum_{k=1}^{n} \sum_{l \neq k} \mathbf{b}_k^T C \mathbf{b}_l. \tag{5.21}$$

However, if we use (5.7b) and (5.18) in (5.15b), we have

$$h(t) = \sum_{j=1}^{n} \left(x_j - \frac{\sigma_j^2 b_{jj}^2}{2} \right) - \frac{1}{2} \sum_{k=1}^{n} \sum_{l \neq k} \rho_{kl} \sigma_k \sigma_l b_{kk} b_{ll}. \tag{5.22}$$

We note from (5.22) that if the variables are independent, $\rho_{kl} = 0$ and we have the independent default approximation (3.4). Continuing to simplify, we have

$$h(t) = \sum_{j=1}^{n} x_j - \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \rho_{kl} \sigma_k \sigma_l b_{kk} b_{ll}, \tag{5.23a}$$

$$= \sum_{j=1}^{n} x_j - \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \mathbf{b}_k^T C \mathbf{b}_l, \tag{5.23b}$$

where we have used (5.20). We note that (5.23b) is a generalization of (5.7b), since if $n = 1$, the sums collapse and $C = \sigma_j^2$. Financially, we see that the default rate h must always contain the effect of the variance of any underlying related random variables. If there is only one variable H_j, then the only effect is through σ_j^2. However, if there is more than one variable, then the effect manifests itself in the covariance matrix C.

Substituting (5.8) into (5.18) and (5.23a), we obtain

$$b'_\rho = -\frac{t^2}{2} \sum_{k=1}^{n} \sum_{l \neq k} \rho_{kl} \sigma_k \sigma_l \tag{5.24a}$$

$$h(t) = -\frac{st^2}{2} + \sum_{j=1}^{n} x_j, \quad s = \sum_{k=1}^{n} \sum_{l=1}^{n} c_{kl} \tag{5.24b}$$

Substituting (5.24b) into (5.15a), we have

$$Q(t) = \exp\left(\frac{st^3}{6} - t \sum_{j=1}^{n} x_j \right). \tag{5.25}$$

Note that $h(t) < 0$, unless

$$t \leq \sqrt{\frac{2}{s} \sum_{j=1}^{n} x_j}.$$

Section 6: Adding Drift

The next complication we add is that of drift, or, in this case, reversion to the mean. We change (5.1a) to the following:

$$dH_j = (r_j - a_{jj}H_j)\,dt + \sigma_j\,dW_j, \tag{6.1a}$$

where r_j and a_{jj} are positive constants. Essentially, (6.1a) says that H_j may oscillate following Brownian motion, but is still drawn to a steady state (mean) r_j/a_{jj}. This choice of diffusion leads to the following forms:

$$g_{j,1}(\mathbf{H}) = r_j - a_{jj}H_j, \qquad g_{j,2}(\mathbf{H}) = \sigma_j. \tag{6.1b}$$

Substituting (6.1b) into (4.5a), we obtain

$$-\frac{\partial Q_j}{\partial t} + \sum_{k=1}^{n}(r_k - a_{kk}x_k)\frac{\partial Q_j}{\partial x_k} + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\rho_{kl}\sigma_k\sigma_l\frac{\partial^2 Q_j}{\partial x_k \partial x_l} = x_j Q_j$$

subject to (4.5b). However, since the only component of $\mathbf{x}$ that appears explicitly is x_j, we may again use the guess in (5.3) to reduce the above to

$$-\frac{\partial Q_j}{\partial t} + (r_j - a_{jj}x_j)\frac{\partial Q_j}{\partial x_j} + \frac{\sigma_j^2}{2}\frac{\partial^2 Q_j}{\partial x_j^2} = x_j Q_j \tag{6.2}$$

subject to (5.4b).

We again use the form in (5.5). Substituting (5.5) into (6.2) and using (5.6), we obtain

$$-\left(-x_j b'_{jj} - b'_{0j}\right)Q_j + (r_j - a_{jj}x_j)(-b_{jj})Q_j + \frac{\sigma_j^2}{2}b_{jj}^2 Q_j = x_j Q_j, \tag{6.3a}$$

$$h_j = x_j - \frac{\sigma_j^2}{2}b_{jj}^2 + b_{jj}(r_j - a_{jj}x_j). \tag{6.3b}$$

Note from (6.3b) that (as in section 5) we never have to solve for b_{0j}! We use (6.3b) later to simplify our algebra; to obtain a solution, we match coefficients of x_j in (6.3a) to obtain

$$\begin{aligned} b'_{jj} + a_{jj}b_{jj} &= 1 \\ b_{jj} &= \frac{1 - e^{-a_{jj}t}}{a_{jj}}. \end{aligned} \tag{6.4}$$

Note that in the limit as $a_{jj} \to 0$, $b_{jj} \to t$, as in (5.8). Also, upon examination of (6.3b), we see that as $t \to \infty$, b_{jj} approaches a constant and hence so does h_j:

$$h_j \to \frac{1}{a_{jj}} \left(r_j - \frac{\sigma_j^2}{2a_{jj}} \right).$$

Thus h_j can go negative in this case as well if $\sigma_j^2 > 2r_j a_{jj}$. Since it is possible for h_j to have an internal minimum, setting the variance below this threshold is necessary to keep h_j positive, but is not sufficient.

Substituting (6.1b) into (4.7a), we obtain

$$-\frac{\partial Q}{\partial t} + \sum_{k=1}^{n} (r_k - a_{kk}x_k) \frac{\partial Q}{\partial x_k} + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \rho_{kl}\sigma_k\sigma_l \frac{\partial^2 Q}{\partial x_k \partial x_l} = Q \sum_{j=1}^{n} x_j \tag{6.5}$$

subject to (4.7b). If we define the diagonal matrix A by

$$a_{kl} = a_{kk}\delta_{kl}, \tag{6.6}$$

where δ_{kl} is the Kronecker delta, then

$$(\mathbf{r} - A^T\mathbf{x}) \cdot \nabla Q = \sum_{j=1}^{n} (r_j - a_{jj}x_j) \frac{\partial Q}{\partial x_j}. \tag{6.7}$$

(The reason for the transpose will become clear in section 7.) Thus (6.5) may be rewritten using (5.12) as

$$-\frac{\partial Q}{\partial t} + (\mathbf{r} - A^T\mathbf{x}) \cdot \nabla Q + \frac{1}{2} \nabla \cdot (C \nabla Q) = Q \sum_{j=1}^{n} x_j. \tag{6.8}$$

We again use (5.15). Substituting (5.15a) into (6.5) and using (5.16a), (5.17), and (6.3b), we have

$$\left\{ b'_\rho + \sum_{j=1}^{n} \left[x_j - \frac{\sigma_j^2}{2} b_{jj}^2 + b_{jj}(r_j - a_{jj}x_j) \right] \right\} Q + \sum_{j=1}^{n} (r_j - a_{jj}x_j)(-b_{jj}Q)$$
$$+ \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} \rho_{jl}\sigma_j\sigma_l b_{jj} b_{ll} Q = Q \sum_{j=1}^{n} x_j,$$

which after cancellation results in (5.21). Substituting (5.21) and (6.3b) into (5.15b), we obtain

$$\begin{aligned} h(t) &= \sum_{j=1}^{n} \left[x_j + b_{jj}(r_j - a_{jj}x_j) \right] - \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \mathbf{b}_k^T C \mathbf{b}_l \\ &= -\frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbf{b}_j^T C \mathbf{b}_k + \sum_{j=1}^{n} \left[x_j + (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_j \right]. \end{aligned} \tag{6.9}$$

Section 7: Adding Coupling

The next complication we add is that of coupling of the drift terms. To do so, we change (6.1a) to the following:

$$dH_j = \left(r_j - \sum_{l=1}^{n} a_{lj} H_l \right) dt + \sigma_j \, dW_j. \tag{7.1a}$$

Essentially, the random process is governed as in section 6, but the mean (steady state) is now determined by the solution of a linear system that includes the effects of all other variables. This choice of diffusion leads to the following forms:

$$g_{j,1}(\mathbf{H}) = r_j - \sum_{l=1}^{n} a_{lj} H_l, \qquad g_{j,2}(\mathbf{H}) = \sigma_j. \tag{7.1b}$$

Substituting (7.1b) into (4.5a), we obtain

$$-\frac{\partial Q_j}{\partial t} + \sum_{k=1}^{n} \left(r_k - \sum_{l=1}^{n} a_{lk} x_l \right) \frac{\partial Q_j}{\partial x_k} + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \rho_{kl} \sigma_k \sigma_l \frac{\partial^2 Q_j}{\partial x_k \partial x_l} = x_j Q_j \tag{7.2}$$

subject to (4.5b). Note that in this case we must include the correlation in the equations for each of the Q_j as all the components of $\mathbf{x}$ appear explicitly in the equation. Again using (5.12) we rewrite (7.2) as

$$-\frac{\partial Q_j}{\partial t} + (\mathbf{r} - A^T \mathbf{x}) \cdot \nabla Q_j + \frac{1}{2} \nabla \cdot (C \nabla Q_j) = (\mathbf{e}_j \cdot \mathbf{x}) Q_j, \tag{7.3}$$

where we have removed the diagonal restriction on A given in (6.6). From (7.2) we see that Q_j is now a function of $\mathbf{x}$ and t, so we replace (5.5) with

$$Q_j(\mathbf{x}, t) = \exp\left(- \sum_{k=1}^{n} b_{kj}(t) x_k - b_{0j}(t) \right), \qquad b_{0j}(0) = b_{kj}(0) = 0, \tag{7.4}$$

where the initial conditions have been chosen to satisfy (4.5b). If we let the kth entry (row) in the column vector $\mathbf{b}_j$ be b_{kj}, then we may rewrite (7.4) as

$$Q_j(\mathbf{x}, t) = \exp\left(-\mathbf{b}_j \cdot \mathbf{x} - b_{0j} \right), \qquad b_{0j}(0) = 0, \qquad \mathbf{b}_j(0) = \mathbf{0}. \tag{7.5}$$

Note that (7.5) is a natural generalization of (5.5) given the definition in (5.19) used in the decoupled case.

We note from (3.3) that with this *ansatz*, the default rate for company j becomes

$$h_j = \mathbf{b}'_j \cdot \mathbf{x} + b'_{0j}. \tag{7.6}$$

As preparatory steps, we note that

$$\begin{aligned}
\frac{\partial Q_j}{\partial x_k} &= -b_{kj} Q_j \\
\nabla Q_j &= -Q_j \mathbf{b}_j, && \text{(7.7a)} \\
\nabla \cdot (C \nabla Q_j) &= -\nabla \cdot (Q_j C \mathbf{b}_j) = -(\nabla Q_j) \cdot (C \mathbf{b}_j) = -(-\mathbf{b}_j Q_j) \cdot (C \mathbf{b}_j) \\
&= \mathbf{b}_j^T C \mathbf{b}_j Q_j, && \text{(7.7b)}
\end{aligned}$$

where in the next-to-last line we have used the fact that Q_j is the only term in the right-hand side of (7.7) that depends on $\mathbf{x}$.

Substituting (7.5) into (7.3) and using (7.7), we obtain

$$-\left(-\mathbf{b}'_j \cdot \mathbf{x} - b'_{0j}\right) Q_j + (\mathbf{r} - A^T \mathbf{x}) \cdot (-\mathbf{b}_j) Q_j + \frac{1}{2} \mathbf{b}_j^T C \mathbf{b}_j Q_j = Q_j (\mathbf{e}_j \cdot \mathbf{x}) \tag{7.8a}$$

$$(\mathbf{b}'_j + A \mathbf{b}_j - \mathbf{e}_j) \cdot \mathbf{x} + b'_{0j} - \mathbf{r} \cdot \mathbf{b}_j + \frac{1}{2} \mathbf{b}_j^T C \mathbf{b}_j = 0, \tag{7.8b}$$

$$h_j = x_j - \frac{1}{2} \mathbf{b}_j^T C \mathbf{b}_j + (\mathbf{r} - A^T \mathbf{x}) \cdot \mathbf{b}_j, \tag{7.9}$$

where we have used (7.6). Note that (7.9) is a generalization of (6.3b) given the definition in (5.19) used in the decoupled case.

Note from (7.9) that (as in sections 5 and 6) we never have to solve for b_{0j}! We use (7.9) later to simplify our algebra; to obtain a solution, we match coefficients of $\mathbf{x}$ in (7.8b) to obtain

$$\mathbf{b}'_j + A \mathbf{b}_j = \mathbf{e}_j. \tag{7.10a}$$

If we construct a matrix B whose jth column is $\mathbf{b}_j$, then we may solve for all the $\mathbf{b}_j$ together:

$$B' + AB = I, \qquad B(0) = O, \tag{7.10b}$$

$$B = B_{\mathrm{h}} + A^{-1} \qquad \Longrightarrow \qquad B'_{\mathrm{h}} + A B_{\mathrm{h}} = O, \qquad B_{\mathrm{h}}(0) = -A^{-1},$$

$$B_{\mathrm{h}}(t) = -A^{-1} e^{-tA},$$

$$B(t) = A^{-1}(I - e^{-tA}). \tag{7.11}$$

We need $B(\infty) = O$, so all the eigenvalues of A must be positive: hence A must be positive definite. Note that (7.11) is a generalization of (6.4) to multiple dimensions.

Again, we see from (7.9) that h_j can become negative, though the calculation is more involved than that in section 6, unless additional restrictions are imposed.

Substituting (7.1b) into (4.7a), we obtain

$$-\frac{\partial Q}{\partial t} + \sum_{k=1}^{n} \left(r_k - \sum_{l=1}^{n} a_{lk} x_l \right) \frac{\partial Q}{\partial x_k} + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \rho_{kl} \sigma_k \sigma_l \frac{\partial^2 Q}{\partial x_k \partial x_l} = Q \sum_{j=1}^{n} x_j, \tag{7.12}$$

and hence (6.8) also holds with our new definition of A.

We again use (5.14), but recall that Q_j now depends on $\mathbf{x}$. As preparatory steps, we note that

$$\nabla Q = \sum_{k=1}^{n} \nabla Q_k \frac{Q}{Q_k} = -Q \sum_{k=1}^{n} \mathbf{b}_k \tag{7.13a}$$

$$\begin{aligned} \nabla \cdot (C \nabla Q) &= -\nabla \cdot \left(QC \sum_{k=1}^{n} \mathbf{b}_k \right) = -(\nabla Q) \cdot \left(C \sum_{k=1}^{n} \mathbf{b}_k \right) \\ &= - \left(-Q \sum_{j=1}^{n} \mathbf{b}_j \right) \cdot \left(C \sum_{k=1}^{n} \mathbf{b}_k \right) \\ &= Q \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbf{b}_j^T C \mathbf{b}_k, \end{aligned} \tag{7.13b}$$

Substituting (5.14) into (6.8) and using (5.16a), (7.9), and (7.13), we have

$$\left\{ b'_\rho + \sum_{j=1}^{n} \left[x_j - \frac{1}{2} \mathbf{b}_j^T C \mathbf{b}_j + (\mathbf{r} - A^T \mathbf{x}) \cdot \mathbf{b}_j \right] \right\} Q + (\mathbf{r} - A^T \mathbf{x}) \cdot Q \sum_{j=1}^{n} (-\mathbf{b}_j) + \frac{1}{2} Q \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbf{b}_j^T C \mathbf{b}_k = Q \sum_{j=1}^{n} x_j.$$

We immediately obtain the same cancellation as previously, so (5.21) holds with our new definition of $\mathbf{b}_j$, and (6.9) holds with our new definition of A.

Section 8: Changing the Variance

In each of the previous sections, it was possible for $h_j(t)$ to become negative under certain conditions. This was because H_j followed a strict Wiener process and so could move in either direction without a preference. In truth, we wish to bound h_j away from zero, so we expect that if H_j approaches zero, the size of the jumps will decrease. Thus, we wish to replace (7.1) with the following form:

$$dH_j = \left(r_j - \sum_{l=1}^{n} a_{lj} H_l \right) dt + \sigma_j F(\mathbf{H})\, dW_j, \qquad F(\mathbf{H}_{\min}) = 0. \tag{8.1}$$

Then with such a postulate, H_j could never decrease beyond some minimum threshold value $H_{\min}$. Essentially, the random process is as in section 7, but the size of the jumps now varies with $\mathbf{H}$.

We begin by letting $F(\mathbf{H}) = \sqrt{H_j}$ in (8.1) to obtain

$$dH_j = \left(r_j - \sum_{l=1}^{n} a_{lj} H_l \right) dt + \sigma_j \sqrt{H_j}\, dW_j. \tag{8.2a}$$

Note that in this case $H_{j,\min} = 0$. For reasons that will become clear momentarily, we set $\rho_{kl} = \delta_{kl}$. Thus the default rates are not correlated, but they are coupled. This choice of diffusion leads to the following forms:

$$g_{j,1}(\mathbf{H}) = r_j - \sum_{l=1}^{n} a_{lj} H_l, \qquad g_{j,2}(\mathbf{H}) = \sigma_j \sqrt{H_j}. \tag{8.2b}$$

Substituting (8.2b) into (4.5a) with $\rho_{kl} = \delta_{kl}$, we obtain

$$-\frac{\partial Q_j}{\partial t} + \sum_{k=1}^{n} \left(r_k - \sum_{l=1}^{n} a_{lk} x_l \right) \frac{\partial Q_j}{\partial x_k} + \frac{1}{2} \sum_{k=1}^{n} \sigma_k^2 x_k \frac{\partial^2 Q_j}{\partial x_k^2} = x_j Q_j \tag{8.3}$$

subject to (4.5b). We like this form because the xs still appear only linearly. If we had introduced a nonzero correlation, we would have had terms that depended on $\sqrt{x_j x_k}$. This would have prevented the *ansatz* in (7.5) from working, and we would have had to solve the entire PDE.

Since we have eliminated the ρ terms, we again use (7.5). As a preparatory step, we note that with (7.5)

$$\frac{\partial^2 Q_j}{\partial x_k^2} = -b_{kj} \frac{\partial Q_j}{\partial x_k} = b_{kj}^2 Q_j. \tag{8.4}$$

Substituting (7.5) into (8.3) and using (7.7a) and (8.4), we obtain

$$-\left(-\mathbf{b}_j' \cdot \mathbf{x} - b_{0j}'\right) Q_j + (\mathbf{r} - A^T\mathbf{x}) \cdot (-\mathbf{b}_j) Q_j + \frac{1}{2}\sum_{k=1}^{n} \sigma_k^2 x_k b_{kj}^2 Q_j = Q_j(\mathbf{e}_j \cdot \mathbf{x}). \tag{8.5}$$

If we define the matrix Y such that

$$y_{kj} = \frac{\sigma_k^2 b_{kj}^2}{2}, \tag{8.6}$$

then we may rewrite the above as

$$\mathbf{b}_j' \cdot \mathbf{x} + b_{0j}' - (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_j + \mathbf{y}_j \cdot \mathbf{x} = (\mathbf{e}_j \cdot \mathbf{x}) \tag{8.7a}$$

$$(\mathbf{b}_j' + A\mathbf{b}_j + \mathbf{y}_j - \mathbf{e}_j) \cdot \mathbf{x} + b_{0j}' - \mathbf{r} \cdot \mathbf{b}_j = 0, \tag{8.7b}$$

$$h_j = x_j - \mathbf{y}_j \cdot \mathbf{x} + (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_j, \tag{8.8}$$

where $\mathbf{y}_j$ is the jth column of Y.

Note from (8.8) that (as in sections 5–7) we never have to solve for b_{0j}! However, in order for us to compute solutions, we will do so. To begin the solution process, we match coefficients of $\mathbf{x}$ and 1 in (8.7b) to obtain

$$\mathbf{b}_j' + A\mathbf{b}_j + \mathbf{y}_j = \mathbf{e}_j, \tag{8.9a}$$

$$B' + AB + Y = I, \qquad B(0) = O, \tag{8.9b}$$

$$b_{0j}' - \mathbf{r} \cdot \mathbf{b}_j = 0. \tag{8.10}$$

Equations (8.9) are analogous to (7.10) with one key difference: they are nonlinear because of the forcing Y. We tried various perturbation schemes to see if we could obtain analytical solutions, but each time the first-order correction terms involved very messy and perhaps insoluble equations. Thus we focused on obtaining numerical solutions as described at the end of this section.

Substituting (8.2b) into (4.7a), we obtain

$$-\frac{\partial Q}{\partial t} + \sum_{k=1}^{n} \left(r_k - \sum_{l=1}^{n} a_{lk} x_l \right) \frac{\partial Q}{\partial x_k} + \frac{1}{2}\sum_{k=1}^{n} \sigma_k^2 x_k \frac{\partial^2 Q}{\partial x_k^2} = Q \sum_{j=1}^{n} x_j. \tag{8.11}$$

Unfortunately, due to the nonlinearity in the ODE system (8.9b), substituting in (5.14) will not result in the sort of cancellation that yielded a simple equation for b_ρ'. However, motivated by (7.5), we let

$$Q(\mathbf{x}, t) = \exp\left(-\mathbf{b}_0 \cdot \mathbf{x} - b_{00}\right), \qquad b_{00}(0) = 0, \qquad \mathbf{b}_0(0) = \mathbf{0}, \tag{8.12a}$$

from which the default rate h becomes

$$h = \mathbf{b}_0' \cdot \mathbf{x} + b_{00}'. \tag{8.12b}$$

Then since the operators on the left-hand sides of (8.3) and (8.11) are identical, upon substitution of (8.12a) into (8.11), we obtain

$$-(-\mathbf{b}_0' \cdot \mathbf{x} - b_{00}')Q + (\mathbf{r} - A^T\mathbf{x}) \cdot (-\mathbf{b}_0)Q + (\mathbf{y}_0 \cdot \mathbf{x})Q = Q\sum_{j=1}^{n} x_j, \tag{8.13}$$

where $\mathbf{y}_0$ is defined in an analogous way to $\mathbf{y}_j$. Solving for h_0, we have

$$h_0 = (\mathbf{1} - \mathbf{y}_0) \cdot \mathbf{x} + (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_0, \tag{8.14}$$

where $\mathbf{1}$ is the n-dimensional vector whose entries are all equal to 1. Note from (8.14) that we never need to solve for b_{00}! However, in order for us to compute solutions, we will do so. Continuing to simplify (8.13), we have

$$\begin{aligned}
\mathbf{b}_0' \cdot \mathbf{x} + b_{00}' - (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_0 + (\mathbf{y}_0 - \mathbf{1} \cdot \mathbf{x}) &= 0 \\
(\mathbf{b}_0' + A\mathbf{b}_0 + \mathbf{y}_0 - \mathbf{1}) \cdot \mathbf{x} + b_{00}' - \mathbf{r} \cdot \mathbf{b}_0 &= 0 && (8.15)\\
\mathbf{b}_0' + A\mathbf{b}_0 + \mathbf{y}_0 &= \mathbf{1}, && (8.16a)\\
b_{00}' - \mathbf{r} \cdot \mathbf{b}_0 &= 0, && (8.16b)
\end{aligned}$$

where we have matched coefficients of $\mathbf{x}$ and 1. Note that equations (8.16) are in the same form as (8.9a) and (8.10).

In order to obtain some numerical results, we specialize to the case where $n = 2$. For completeness, we write down the equations explicitly. Equations (8.9b) and (8.10) become

$$\begin{aligned}
b_{11}' + a_{11}b_{11} + a_{12}b_{21} + \frac{\sigma_1^2 b_{11}^2}{2} &= 1, && (8.17a)\\
b_{21}' + a_{21}b_{11} + a_{22}b_{21} + \frac{\sigma_2^2 b_{21}^2}{2} &= 0, && (8.17b)\\
b_{01}' - r_1 b_{11} - r_2 b_{21} &= 0, && (8.17c)\\
b_{12}' + a_{11}b_{12} + a_{12}b_{22} + \frac{\sigma_1^2 b_{12}^2}{2} &= 0, && (8.18a)\\
b_{22}' + a_{21}b_{12} + a_{22}b_{22} + \frac{\sigma_2^2 b_{22}^2}{2} &= 1, && (8.18b)\\
b_{02}' - r_1 b_{12} - r_2 b_{22} &= 0, && (8.18c)
\end{aligned}$$

while equations (8.16) become

$$\begin{aligned}
b_{10}' + a_{11}b_{10} + a_{12}b_{20} + \frac{\sigma_1^2 b_{10}^2}{2} &= 1, && (8.19a)\\
b_{20}' + a_{21}b_{10} + a_{22}b_{20} + \frac{\sigma_2^2 b_{20}^2}{2} &= 1, && (8.19b)\\
b_{00}' - r_1 b_{10} - r_2 b_{20} &= 0. && (8.19c)
\end{aligned}$$

As mentioned above, equations (8.17c), (8.18c), and (8.19c) are not necessary, but make the Maple calculations more straightforward.

For the calculations, we took the following parameter values as constants:

$$a_{11} = 1, \quad a_{22} = 1, \quad r_1 = 0.05, \quad r_2 = 0.06, \quad \sigma_1 = 1/4, \quad \sigma_2 = 1/3, \quad x_2 = 0.03, \tag{8.20a}$$

$$a_{12} = a_{21} = -0.4, \tag{8.20b}$$

and then varied the value of x_1. We begin with the case where $x_1 = 0.4$, which means that initially the default rate is 40%. This might correspond to a tech startup, but even this value seems extraordinarily large for a real company.

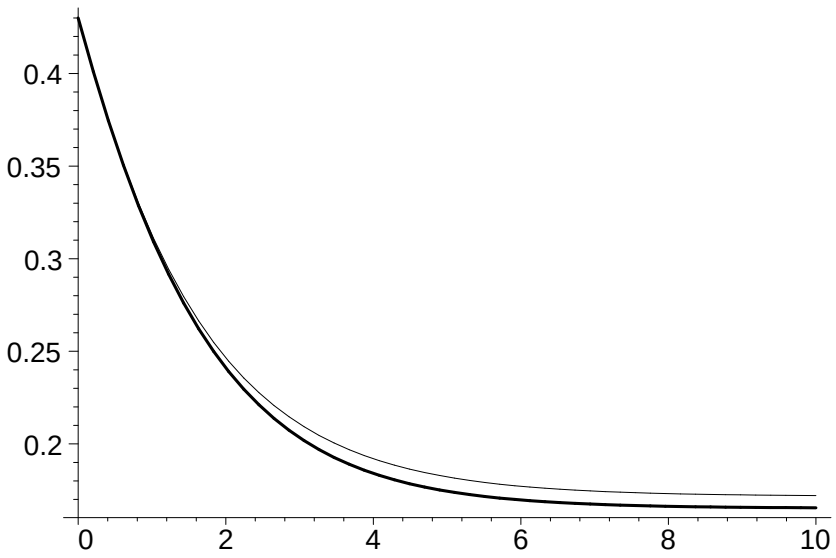

Figure 8.1. Comparison of h (thick line) and $h_1 + h_2$ (thin line) *vs.* t for parameters in (8.20) and $x_1 = 0.4$.

In figure 8.1 we show a comparison of h and $h_1 + h_2$ for this case. Note that the default rate decays from the high value to a more moderate one. Here the steady state is rather large (around 16%). The steady state can easily be adjusted by changing the parameters; however, we did not have enough time at the workshop to do so. As expected, we overestimate h by adding h_1 and h_2. Figure 8.2 shows the size of the overestimate. Note that as $t \to \infty$, it asymptotes to a finite nonzero value.

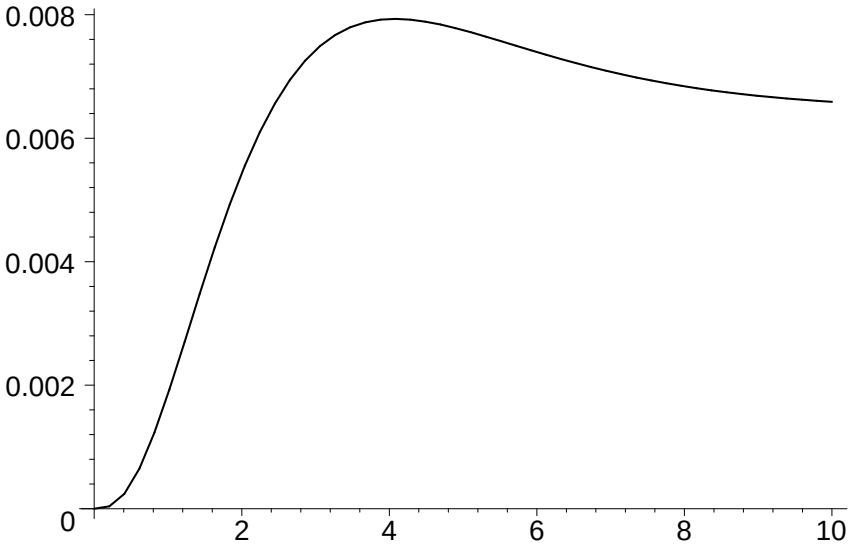

Figure 8.2. $h_1 + h_2 - h$ *vs.* t for parameters in (8.20) and $x_1 = 0.4$.

Next we consider the case where $x_1 = 0.14$, which is very close to the steady state. Therefore in figure 8.3 you see a bump in the rate as it shoots up briefly and then decays down to the steady state. This could be a model for a temporary crisis that may affect a company's ability to pay bonds. The error looks very large, but this is an artifact of the small scale on the y-axis. We graph the actual error in figure 8.4. Note that the graph is very close in size and shape to the previous case.

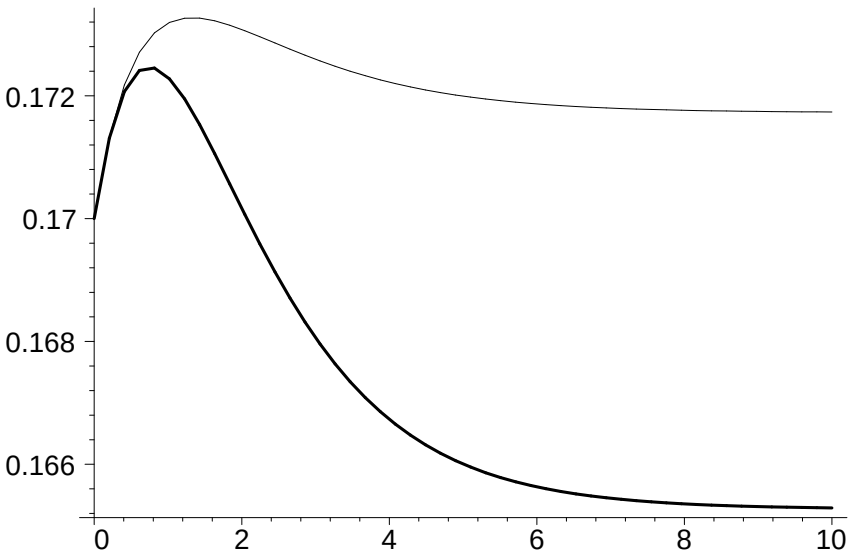

Figure 8.3. Comparison of h (thick line) and $h_1 + h_2$ (thin line) *vs.* t for parameters in (8.20) and $x_1 = 0.14$.

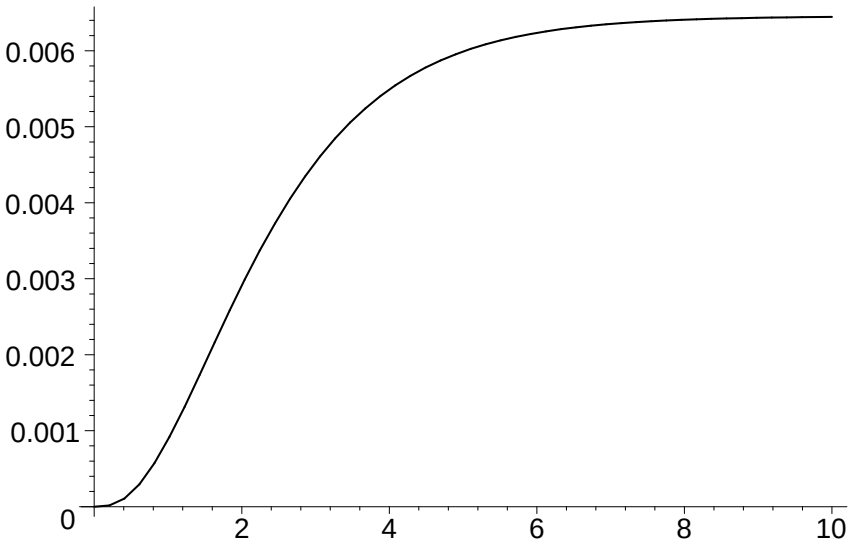

Figure 8.4. $h_1 + h_2 - h$ *vs.* t for parameters in (8.20) and $x_1 = 0.14$.

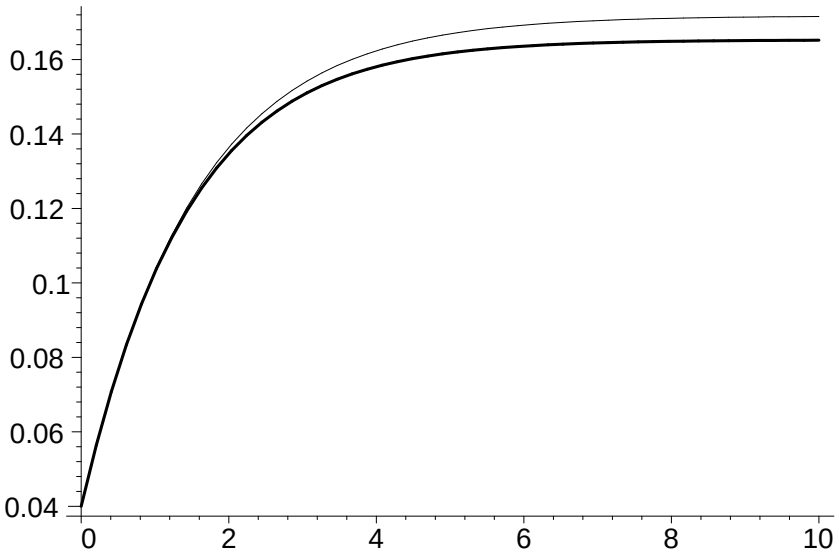

Figure 8.5. Comparison of h (thick line) and $h_1 + h_2$ (thin line) *vs.* t for parameters in (8.20) and $x_1 = 0.01$.

Lastly we consider the case where $x_1 = 0.01$, which corresponds to a very steady company. However, it is coupled to a riskier company. Therefore in figure 8.5 you see growth in the rate to the steady state. Again the error is roughly the same as in the previous cases, as shown in figure 8.6.

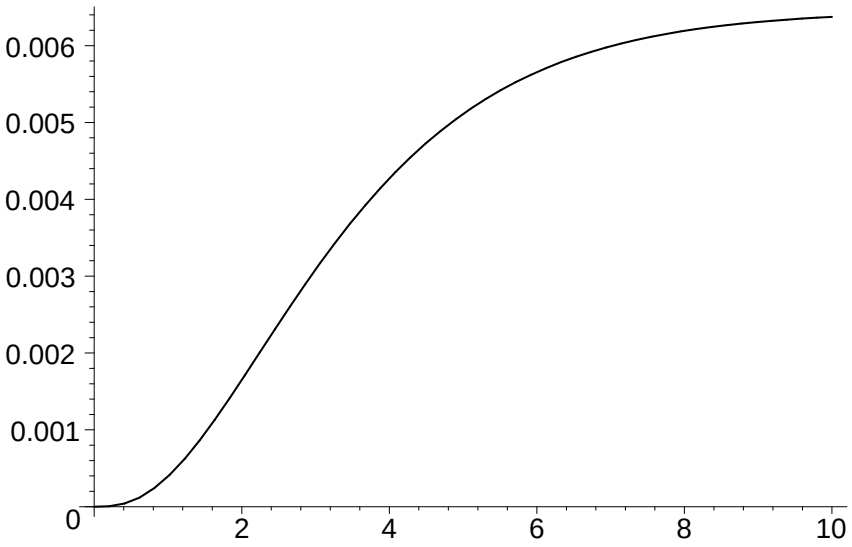

Figure 8.6. $h_1 + h_2 - h$ *vs.* t for parameters in (8.20) and $x_1 = 0.01$.

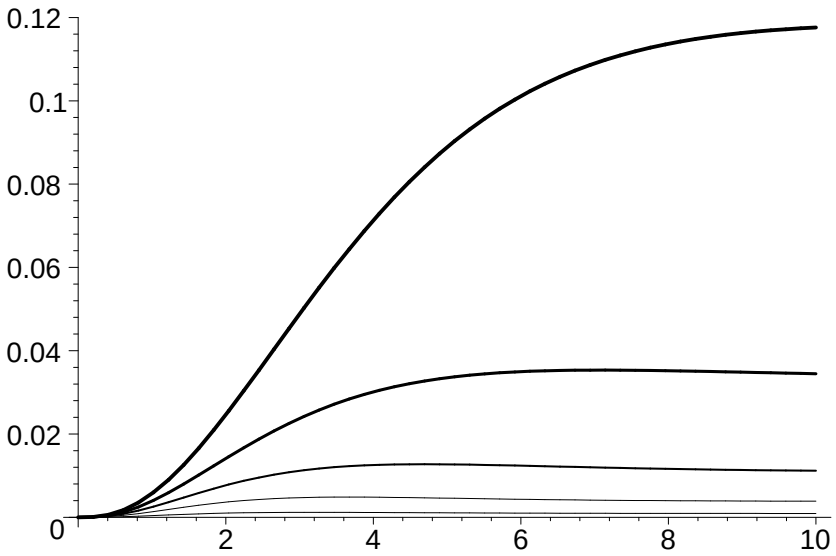

Figure 8.7. $h_1 + h_2 - h$ *vs.* t for parameters in (8.20a) and $x_1 = 0.4$. In decreasing order of thickness: $a_{12} = -0.9$, -0.7, -0.5, -0.3, and -0.1.

Lastly, we note from (8.1) that since $\rho_{12} = 0$, the coupling between the two companies is given by the size of a_{12} (since A is symmetric). If $a_{12} = 0$, the companies are totally independent because we have taken $\rho_{12} = 0$. Therefore, in figure 8.7 we show a plot of the first case keeping the parameters in (8.20a) with $x = 0.4$. However, we let a_{12} vary. We note that as a_{12} gets smaller, the difference gets smaller as the coupling weakens.

Section 9: Extensions

We present briefly some other models considered at the workshop. Each has the property that the equations they generate can be transformed into ODEs *via* the substitutions (7.5) and (8.12a). First we adjust (8.2a) to include a minimum tolerance for each of the variables:

$$dH_j = \left(r_j - \sum_{l=1}^{n} a_{lj} H_l \right) dt + \sigma_j \sqrt{H_j - H_{j,\min}}\, dW_j. \tag{9.1}$$

If we still keep $\rho_{kl} = \delta_{kl}$, then the only change to our work in section 8 is that x_j gets changed to $x_j - x_{j,\min}$ *in the diffusion term only.* Thus (8.5) becomes

$$-\left(-\mathbf{b}'_j \cdot \mathbf{x} - b'_{0j}\right) Q_j + (\mathbf{r} - A^T\mathbf{x}) \cdot (-\mathbf{b}_j) Q_j + \frac{1}{2} \sum_{k=1}^{n} \sigma_k^2 (x_k - x_{k,\min}) b_{kj}^2 Q_j = Q_j (\mathbf{e}_j \cdot \mathbf{x}), \tag{9.2}$$

so (8.7) and (8.8) become

$$\begin{aligned} \mathbf{b}'_j \cdot \mathbf{x} + b'_{0j} - (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_j + \mathbf{y}_j \cdot (\mathbf{x} - \mathbf{x}_{\min}) &= (\mathbf{e}_j \cdot \mathbf{x}) \\ (\mathbf{b}'_j + A\mathbf{b}_j + \mathbf{y}_j - \mathbf{e}_j) \cdot \mathbf{x} + b'_{0j} - \mathbf{r} \cdot \mathbf{b}_j - \mathbf{y}_j \cdot \mathbf{x}_{\min} &= 0, \end{aligned} \tag{9.3a}$$

$$h_j = x_j - \mathbf{y}_j \cdot (\mathbf{x} - \mathbf{x}_{\min}) + (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_j. \tag{9.3b}$$

The coefficient of $\mathbf{x}$ is the same in (9.3a) and (8.7b), so equations (8.9) hold and (8.10) is replaced by

$$b'_{0j} - \mathbf{r} \cdot \mathbf{b}_j - \mathbf{y}_j \cdot \mathbf{x}_{\min} = 0. \tag{9.4}$$

Following the same arguments for Q, we see that (8.13) becomes

$$\begin{aligned} \mathbf{b}'_0 \cdot \mathbf{x} + b'_{00} - (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_0 + \mathbf{y}_0 \cdot (\mathbf{x} - \mathbf{x}_{\min}) &= \mathbf{1} \cdot \mathbf{x} \\ (\mathbf{b}'_0 + A\mathbf{b}_0 + \mathbf{y}_0 - \mathbf{1}) \cdot \mathbf{x} + b'_{00} - \mathbf{r} \cdot \mathbf{b}_0 - \mathbf{y}_0 \cdot \mathbf{x}_{\min} &= 0, \end{aligned} \tag{9.5a}$$

so (8.14) becomes

$$h_0 = \mathbf{1} \cdot \mathbf{x} - \mathbf{y}_0 \cdot (\mathbf{x} - \mathbf{x}_{\min}) + (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_0. \tag{9.5b}$$

The coefficient of $\mathbf{x}$ is the same in (9.5a) and (8.15), so equation (8.16a) holds and (8.16b) is replaced by

$$b'_{00} - \mathbf{r} \cdot \mathbf{b}_0 - \mathbf{y}_0 \cdot \mathbf{x}_{\min} = 0. \tag{9.6}$$

Another model proposed was to let

$$dH_j = \left(r_j - \sum_{l=1}^{n} a_{lj} H_l \right) dt + \sigma_j \sqrt{H_m - H_{m,\min}}\, dW_j, \tag{9.7}$$

for some *particular* value of m. In other words, the size of the random walk term is controlled by just one of the failure rates H_m. This may be a good assumption for a set of subsidiaries of company m, so the fortunes of company m directly affect all the others, but the others don't affect one another very much. However, the model is defective in that for $j \neq m$, there is no mechanism to keep H_j from going negative.

In this case, we can include the correlation, so the analysis to follow is in section 7. Essentially the $\sqrt{H_m - H_{m,\min}}$ term acts like a constant in the covariance summation, so (7.3) becomes

$$-\frac{\partial Q_j}{\partial t} + (\mathbf{r} - A^T\mathbf{x}) \cdot \nabla Q_j + \frac{x_m - x_{m,\min}}{2} \nabla \cdot (C \nabla Q_j) = (\mathbf{e}_j \cdot \mathbf{x}) Q_j,$$

so (7.8) and (7.9) become

$$\mathbf{b}_j' \cdot \mathbf{x} + b_{0j}' - (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_j + \frac{1}{2}(\mathbf{e}_m \cdot \mathbf{x} - x_{m,\min}) \mathbf{b}_j^T C \mathbf{b}_j = \mathbf{e}_j \cdot \mathbf{x}$$

$$\left(\mathbf{b}_j' + A\mathbf{b}_j - \mathbf{e}_j + \frac{1}{2} \mathbf{b}_j^T C \mathbf{b}_j \mathbf{e}_m \right) \cdot \mathbf{x} + b_{0j}' - \mathbf{r} \cdot \mathbf{b}_j - \frac{x_{m,\min}}{2} \mathbf{b}_j^T C \mathbf{b}_j = 0, \tag{9.8a}$$

$$h_j = x_j - \frac{1}{2}(\mathbf{e}_m \cdot \mathbf{x} - x_{m,\min}) \mathbf{b}_j^T C \mathbf{b}_j + (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_j. \tag{9.8b}$$

Though from (9.8b) we see that we don't need to compute b_{0j} to obtain h_j, our experience with nonlinear systems tells us that h_j will be easier to compute if we do by matching the coefficients of $\mathbf{x}$ and 1 in (9.8a):

$$\mathbf{b}_j' + A\mathbf{b}_j + \frac{1}{2} \left(\mathbf{b}_j^T C \mathbf{b}_j \right) \mathbf{e}_m = \mathbf{e}_j, \tag{9.9a}$$

$$b_{0j}' - \mathbf{r} \cdot \mathbf{b}_j - \frac{x_{m,\min}}{2} \mathbf{b}_j^T C \mathbf{b}_j = 0. \tag{9.9b}$$

Since we now have introduced nonlinearity into our problem, we will use the *ansatz* in (8.12a) for Q. Also, since the operators on the left-hand sides of (7.2) and (7.12) are identical, as in the first model in this section we simply replace $\mathbf{e}_j$ by $\mathbf{1}$ in (9.8) and (9.9) to obtain

$$h_0 = \mathbf{1} \cdot \mathbf{x} - \frac{1}{2}(\mathbf{e}_m \cdot \mathbf{x} - x_{m,\min}) \mathbf{b}_0^T C \mathbf{b}_0 + (\mathbf{r} - A^T\mathbf{x}) \cdot \mathbf{b}_0. \tag{9.10}$$

$$\mathbf{b}_0' + A\mathbf{b}_0 + \frac{1}{2} \left(\mathbf{b}_0^T C \mathbf{b}_0 \right) \mathbf{e}_m = \mathbf{1}, \tag{9.11a}$$

$$b_{00}' - \mathbf{r} \cdot \mathbf{b}_0 - \frac{x_{m,\min}}{2} \mathbf{b}_0^T C \mathbf{b}_0 = 0. \tag{9.11b}$$

The final model proposed was for a two-company system: there we let

$$dH_j = \left(r_j - \sum_{l=1}^{2} a_{lj} H_l \right) dt + \sigma_j \sqrt{H_m - H_{m,\min}}\, dW_j, \qquad j \neq m. \tag{9.12}$$

In other words, the size of the random walk term is controlled by the behavior of the other company's failure rate. This has a nice symmetry property, and may be a good assumption for a pair of companies who depend nearly exclusively on one another for business. However, the model is defective in that there is no mechanism to keep H_j from going below $H_{j,\min}$, which from the square root would then push the variables into the complex plane.

In this case, we cannot include the correlation due to the previously mentioned cross-term problem, so the analysis to follow is that of the first model given above. Here (9.2) is replaced by

$$-\left(-\mathbf{b}'_j \cdot \mathbf{x} - b'_{0j}\right) Q_j + (\mathbf{r} - A^T\mathbf{x}) \cdot (-\mathbf{b}_j) Q_j + \frac{1}{2}\sum_{k=1}^{2} \sigma_k^2 (x_m - x_{m,\min}) b_{kj}^2 Q_j = Q_j(\mathbf{e}_j \cdot \mathbf{x}), \quad k \neq m. \tag{9.13}$$

Therefore, if we replace the definition of the matrix Y in (8.6) with

$$y_{mj} = \frac{\sigma_k^2 b_{kj}^2}{2}, \qquad m \neq k, \tag{9.14}$$

then we can use the $\mathbf{y}_j$ notation and (9.3)–(9.6) hold.

Section 10: Conclusions and Further Research

In sections 1 and 2 we introduced the concept of multi-name credit derivatives. In section 3 we tried a very simple model for a joint default rate, but found that the model led to uncorrelated companies as $t \to \infty$: a result not seen in the real world. In section 4 we introduced the Feynman-Kac formula to relate the model for a random process to an underlying PDE that can be solved to obtain the survivor probabilities. Sections 5–9 consisted of slowly increasing the complexity of the underlying random process from a simple Wiener process to include drift, coupling, nonconstant variance, and minimum tolerances.

Numerical results for the case of two companies suggest that the models derived in Sections 8 and 9 may be useful in pricing of multi-name credit derivatives. It would be desirable to test the validity of these models against historical data to determine whether they could actually be used. A more general theory for the evolution of correlated failure rates of a basket of names is still lacking, but some ideas about what is needed may be inferred from the work in this report.

Nomenclature

In the manuscript, boldface indicates a vector where the componenets are the italic letters with subscript j. The equation number where a particular quantity first appears is listed, if appropriate.

A: matrix of constants a_{jk} modeling relationship between H's in the drift term (6.1a).
$B(t)$: matrix of functions $b_{jk}(t)$ in the solution *ansatz*.
C: covariance matrix (5.11).
$D(t)$: discount factor for time t (2.4).
dW: Wiener process (5.1a).
$\mathbf{e}_j$: jth standard unit normal vector (5.19).
$F(\mathbf{H})$: function to keep $\mathbf{H}$ from jumping over to negative values (8.1).
$f(t)$: instantaneous forward rate at time t (3.1).
$G(\mathbf{H})$: arbitrary function (4.2).
$g(\mathbf{H})$: arbitrary function, variously defined (4.1a).
$H_j(t)$: random variable representing the default rate for company j (4.1a).
$h(t)$: default rate at time t (3.2).
i: indexing variable referring to time period.
j: indexing variable referring to company (2.5).
k: indexing variable referring to company (4.1b).
l: indexing variable referring to company (4.2).
M: number of insured time periods.
m: variable referring to company (9.7).
n: number of companies (2.8).
P: probability (2.1).
$\mathcal{Q}$: random variable corresponding to Q.
$Q(t)$: survival probability distribution function (2.1).
r: positive constant in drift process for H (6.1a).
s: positive constant representing correlation (5.24a).
t: time from contract sale.
V: value of contract (2.3).
w: percent of principal recoverable in bankruptcy proceedings.
X_j: indicator variable for default of company j (3.6).
$\mathbf{x}$: vector of initial default rates (4.3).
Y: matrix of nonlinear terms (8.6).
ξ: dummy variable.
ρ: correlation coefficient (3.8).
σ_j: standard deviation related to company j (3.8).
τ: time of default (2.1).
ϕ_i: insurance premium at time t_i.

Other Notation

- c: as a superscript on Q, used to indicate a conditional probability.
- f: as a superscript on Q, used to indicate a probability where one of the issuers defaults first (2.6).
- h: as a subscript, used to indicate a homogeneous solution.
- min: as a subscript, used to indicate a minimum threshold value (8.1).
- ρ: as a subscript, used to indicate the part of the solution that depends on ρ (5.14).
- 0: as a subscript, used to indicate a single-company contract (2.1) or joint distribution (8.12a).
- +: as a subscript, used to indicate a positive quantity.
- $'$: used to indicate a dummy variable.

References

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[2] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*, 5th ed. Berlin: Springer-Verlag, 1998.