WRINKLING OF PAPER LABELS ON BEER BOTTLES

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Abstract

Two different types of wrinkles in beer bottle labels are studied namely wrinkling due to the labeling process and wrinkles due to expansion of the paper as consequence of the moisture in the labels.

Four problems are considered, namely, the geometry of the labelling process, the formation of wrinkles related to physical qualities of the paper, the spreading of the glue and removal of the labels.

An optimal speed was found for the labelling process. The group evaluated the importance of the distance between the glue strips both for wrinkle constraint and removal of the labels. The group also investigated the influence of the angle between the preferential expansion direction of the paper and the glue strips and showed that the glue strips should be perpendicular to the fibres in the paper.

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1 Introduction

South African Breweries (SAB) Ltd produces thousands of beer bottles in their factories per hour. The process may be summarised as follows: clean beer bottles are filled with beer, sealed, pasteurised, labelled and then crated before distribution to retailers. After consumption, the empty bottles are returned to SAB Ltd for recycling: the bottles are uncrated, washed and rid of old labels, inspected and rewashed or discarded if necessary. SAB Ltd encounters difficulties with the labelling process. Two types of defects have been reported:

1. Parts of the label may not adhere to the bottle near the centre of the bottle.

2. Long horizontal wrinkles may appear in a humid environment.

Although these defects do not jeopardise the quality of the beer, they may be hugely detrimental to the marketing of the product. SAB Ltd asked the Study Group to investigate these difficulties.

The group focused mainly on three aspects of the problem, presented here in the order they are encountered during production:

1. Optimal speed for the labelling process

   The labelling of the beer bottles may be divided into several stages. Practically, the back of the beer labels is covered with strips of glue. The labels are then grabbed by two metal pegs and placed in front of the bottles. Sponges then force the label onto the bottle surface and a series of brushes flatten it. Several thousands of bottles are processed every hour. The group investigated the limits imposed by the geometry on the processing speed. If bottles are labelled too fast, the glue will not spread as expected and the area of the label that can not be reached by the brushes will increase. Calculating an optimal production speed will therefore limit the formation of both types of wrinkles. These aspects will be detailed in Section 3.

2. Paper physics and wrinkling formation

   The physical properties of paper labels are other potential key elements in the formation of wrinkles. Combined with the geometry of the glue strips, the properties of the paper may favour or prevent the wrinkling of the label. This aspect of the problem is developed in Section 4 and provides the distance necessary between two glue strips and the best possible orientation.

3. Spreading of the glue

   The spreading of the glue is studied in Section 5. This will provide a relation between the force applied by the brushes, the geometry of the glue strips and labels and the time necessary for a complete spreading of the glue strip. Coupled with the results of Section 3, this would provide an approximation of the number of brushes necessary for the labelling process.

Finally, the removal of the labels will be briefly considered in the last section. When possible results will be compared with the conclusions of the Australia/New Zealand Study Group (MISGAU) report [1, 2] that dealt with the labelling of wine bottles.

2 Results of the MISGAU Study Group

The long horizontal wrinkles between glue strips, due to absorption of water by the paper which causes hygroscopic expansion of the label, was studied in a Study Group held in Australia in 1996 (MISGAU 1996) in the context of wine bottle labelling. Most of the findings in the Australian report are of relevance to this problem. The significant differences between the two industries are the following:

- Wine bottles are stored in a dry place after labelling. This facilitates the glue and label to dry before shipping. In contrast, beer bottles are at outlets sometimes within hours of being labelled. Thus the glue and paper do not have time to dry completely. The Australian report suggests that changes in wrinkling are significant in the time that drying occurs (about five days in a dry environment).

- Beer bottles are recycled and thus the labels need to be removed in a cost efficient manner. In the wine industry bottles are not recycled and thus the industry could move away from water based glue. Here this is not a viable option.

The Australian MISG found that the water in the glue is more than sufficient to completely saturate the label. Since paper is made up of fibres which have a preferred alignment, the water causes the paper to expand up to ten times more in the direction transverse to the fibres than in the direction along the fibre lengths. The following suggestions were made concerning the application of the glue:

- Glue should continue to be applied in glue strips. The spaces between the strips promote drying of the glue and they allow compressed air to escape, thereby avoiding “wallpapering bubbles”.
- Glue strips should be perpendicular (or transverse) to the fibres to reinforce the paper against expansion. The fibres in the labels are horizontal and therefore the glue strips should run vertically.
- The continuity of glue strips in the longitudinal direction should be broken by cuts in order to reduce the effects of longitudinal glue tension.

They also suggested experimenting with labels that have impermeable coatings on both faces. At present only the printed side is varnished. If the underside is impermeable the paper will be less likely to wrinkle but it will not assist the glue drying, so that glues with lower water content may be needed.

3 Geometry of brush-bottle set-up

In this section we analyse the geometry of the brushes used to flatten the labels on the bottle: the objective is to derive an expression for the maximum rate at which bottles can be processed and determine the maximum label size possible.

3.1 Production speed

Figure 1 shows the geometry of a bottle in contact with the brushes. At time $t = 0$, the bottle of radius R gets in touch with the brush of length L. From this moment, the brush starts bending and it applies pressure on the bottle and the label. The contact is lost at time $t = t_f$. The angular positions of the brush at times $t = 0$, t and $t = t_f$ are denoted θ_0, θ and θ_f respectively. The brush touches the next bottle at time $t = t_1$. Note

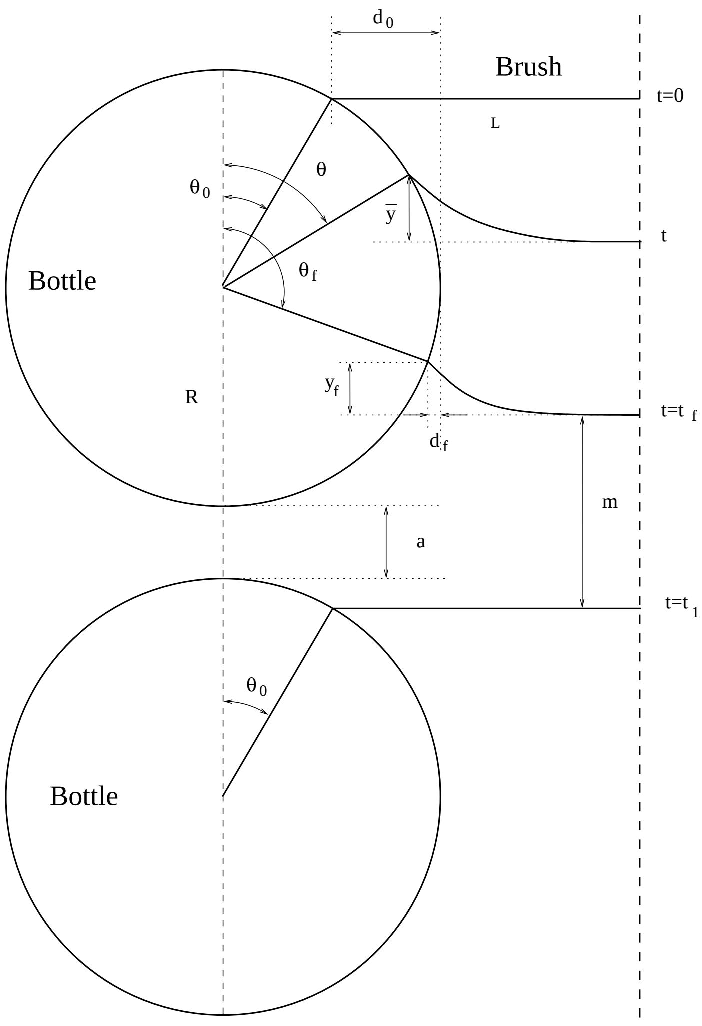


Figure 1: Geometry of brush on beer bottle.

the brushes keep a constant length during the process. This was modified in Figure 1 to facilitate the definition of geometrical parameters.

Straightforward calculations provide the following formulae:

$$\theta_0 = \arcsin\left(\frac{R - d_0}{R}\right) , \quad \theta_f = \frac{\pi}{2} + \arccos\left(\frac{R - d_f}{R}\right) .$$

The brushes are bending during the process as seen experimentally, so there is no simple way to express the distance between the tip of the brush and the horizontal relaxed position $\bar{y}$ as a function of the angle θ. If the brushes are approximated straight during the complete process, $\bar{y}$ may be calculated as

$$\begin{aligned}\bar{y}(\theta) &= \sqrt{L^2 - (R - d_0 + L - R\sin\theta)^2} \qquad (1)\\ &= \sqrt{(R\sin\theta + d_0 - R)\,(2L + R - d_0 - R\sin\theta)} \qquad (2)\end{aligned}$$

and

$$y_f = \frac{L}{R+L}\sqrt{2(R+L)d_0 - d_0^2}\ . \qquad (3)$$

The velocity at which bottles are processed may be expressed as

$$v = \frac{n}{3600}(a + 2R)\ , \qquad (4)$$

where n is the number of bottles processed every hour. If the brushes are to perform as they are intended to, they must be back in the horizontal position when they get in contact with the following bottle, as shown in Figure 1. It takes them the time T to get back in position, during which the bottles are covering the distance m that may be calculated as

$$m = a + 2R - y_f - \sqrt{2Rd_0 - d_0^2} - \sqrt{2Rd_f - d_f^2}\ .$$

Equation (4) can then also be written

$$v = \frac{n}{3600}(a + 2R) = \frac{m}{T} = \frac{a + 2R - y_f - \sqrt{2Rd_0 - d_0^2} - \sqrt{2Rd_f - d_f^2}}{T}\ . \qquad (5)$$

The optimal production speed (in bottles per hour) of the bottles is then

$$n_{optimal} = 3600\frac{a + 2R - y_f - \sqrt{2Rd_0 - d_0^2} - \sqrt{2Rd_f - d_f^2}}{T(a+2R)}\ . \qquad (6)$$

There are three unknowns in (6), namely T, y_f and d_f. If the brushes remain approximately straight, equation (6) can be adapted using formula (3) and determining d_f geometrically:

$$n_{optimal} = 3600\frac{a + 2R - \sqrt{2Rd_0 - d_0^2} - \sqrt{2(R+L)d_0 - d_0^2}}{T(a+2R)}\ . \qquad (7)$$

This approximation may not be too bad in the case of the stiff brushes, taking into consideration that d_0 is small compared to the length of the brushes L and the diameter of the bottles. The time T necessary for the brush to get back to the horizontal position is the only unknown remaining. This aspect will now be discussed.

3.2 Relaxation time

The relaxation time T could be measured using a high speed camera when the labelling machine is only producing a low number of bottles per hour. The time T could also be estimated from standard equations. The Euler-Bernoulli beam equation governs the evolution of a beam with time:

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0 \ , \tag{8}$$

where $y(x,t)$ is the distance from the horizontal position and A, ρ, E and I denote the cross-section area, density, Young modulus and the second moment of inertia (or area moment of inertia) respectively. If the beams in a brush have the same behaviour as a single beam, this means neglecting the interaction between the beams and the damping caused by the interaction, the oscillation time of the beam may be calculated from this governing equation. The relaxation time may first be estimated by non-dimensionalising (8). The scales are defined by

$$x = Lx' \ , \quad y = y_f y' \ , \quad t = \tau t' \ ,$$

where the quantities with primes define the non-dimensional parameters. Dropping the primes immediately, equation (8) becomes

$$\frac{\rho A L^4}{\tau^2 EI} \frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = 0 \ . \tag{9}$$

The oscillation time τ can then be approximated as

$$\frac{\rho A L^4}{\tau^2 EI} = 1 \Longrightarrow \tau = \sqrt{\frac{\rho A L^4}{EI}} \ . \tag{10}$$

The beam must only travel a quarter of the oscillation to come back to a horizontal position. A good order of magnitude for the time T used in the previous section is therefore

$$T \approx \frac{\tau}{4} = \frac{1}{4}\sqrt{\frac{\rho A L^4}{EI}} \ . \tag{11}$$

A more detailed study of the governing equation (8) shows that the frequencies of oscillations for the beam are given by [3, 4]

$$\omega_n = \Theta_n^2 \sqrt{\frac{EI}{\rho A L^4}} \tag{12}$$

where Θ_n are solutions of the equation

$$\cosh \Theta_n \cos \Theta_n = -1 \ .$$

This equation leads to

$$\Theta_1 = 1.9 \ , \quad \Theta_2 = 4.7 \ , \quad \Theta_3 = 7.9 \ , \quad \Theta_4 = 11.0 \ .$$

Only considering the first mode, the oscillation period is now:

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\Theta_1^2}\sqrt{\frac{\rho A L^4}{EI}}$$

and therefore

$$T = \frac{1}{4}\frac{2\pi}{\Theta_1^2}\sqrt{\frac{\rho A L^4}{EI}} \approx \frac{1}{2}\sqrt{\frac{\rho A L^4}{EI}} \ . \tag{13}$$

This value is consistent with the order of magnitude calculated above and provides a good estimation for time T when considering a single beam. The brush is in contact with the bottle during the time

$$t_f = \frac{3600}{n} - T \ . \tag{14}$$

This value will be required in Section 5.

The beams will interact and this value will be modified. In the longer term, the bristles of the brush will wear and will not come back to a horizontal position. These aspects will now be considered.

3.3 Brush modelling

From observations, we consider the brush as a simple harmonic oscillator that is critically damped. The equation that describes such oscillations are given by the ordinary differential equation

$$\ddot{x} + \nu\dot{x} + \frac{\nu^2}{4}x = 0 \ ,$$

where the overhead dot denotes differentiation with respect to t and x is the distance from the neutral position. The general solution of this equation is

$$x(t) = c_1 e^{-\nu t/2} + c_2 t e^{-\nu t/2} \ .$$

The two constants c_1 and c_2 are found from the initial conditions $x(0) = y_f$ and $\dot{x}(0) = V_0$. The value of V_0 may be calculated as follows when the brushes are assumed to remain straight. From Figure 1,

$$R\cos\theta_0 + R\cos\theta + y - vt = 0 \ .$$

From the time derivative of this formula and equation (2), and assuming that the bristles on the brush remain straight, we find that

$$y\dot{y} = (R - d + L - R\sin\theta)R\cos\theta\frac{\dot{y} - v}{R\sin\theta} \ .$$

When $y = y_f$ and $\theta = \theta_f$ we have $\dot{y} = v_0$ and thus

$$v_0 = \frac{v\left(-RL + Ld_0 - L^2 + R\sqrt{\frac{L\left(R+L-\sqrt{2\,Rd_0+2\,Ld_0-{d_0}^2}\right)}{R+L}}\right)}{\left((L+R)\sqrt{\frac{L\left(R+L-\sqrt{2\,Rd_0+2\,Ld_0-{d_0}^2}\right)}{R+L}} - RL + Ld_0 - L^2\right)} \ .$$

This leads to $v_0 \approx 1.2v$ when $L = R = 5$cm and $d = 1cm$.

The speed v_0 is the initial speed of the tip of the brush with respect to the centre of the bottle. Thus the initial speed of the tip of the brush with respect to the root of the brush is $V_0 = v_0 - v$. Therefore $c_1 = y_f$ and $c_2 = V_0 + \nu y_f/2$ and the solution to the initial value problem is thus

$$x(t) = y_f e^{-\nu t/2} + \left(v_0 - v + \frac{\nu y_f}{2}\right) t e^{-\nu t/2} \ .$$

We need to express ν in terms of T. We consider $t = k/\nu$, then

$$x(k/\nu)/y_f = (1 + k/2)e^{-k/2} + vk/(y_f \nu)e^{-k/2} .$$

The second term is probably small and definitely negative, so

$$x/y_f < (1 + k/2)e^{-k/2} .$$

Figure 2 is a graph of the right hand side of this equation. We see that for $k = 8$, $x < 0.1y_f$ and therefore close to the relaxed position. We thus take $T = 8/\nu$ or $\nu = 8/T$ which gives

$$x(t) = y_f e^{-4t/T} + \left(v_0 + \frac{4y_f}{T}\right) te^{-4t/T} .$$

This equation will now be used to study where the brush first touches the bottle.

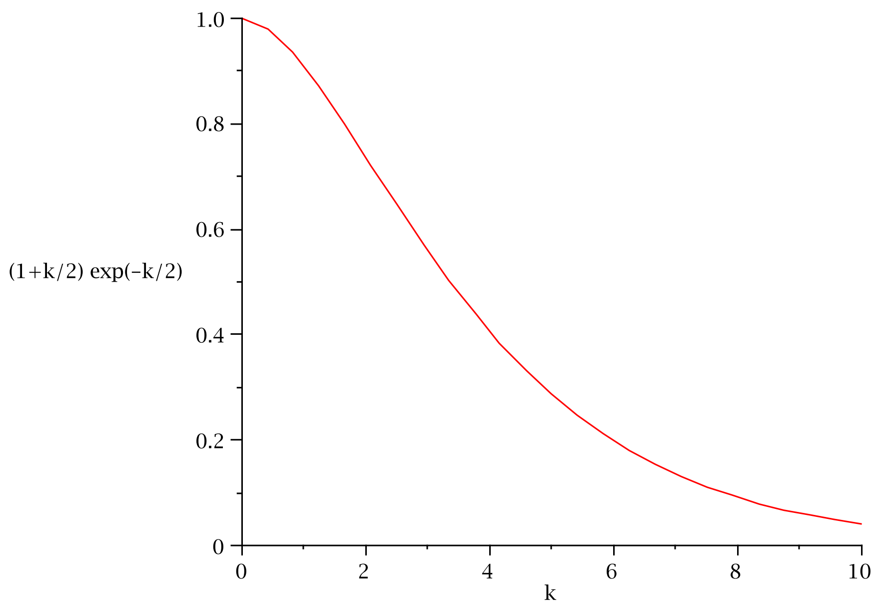


Figure 2: Graph of $(1 + k/2)e^{-k/2}$ against k.

3.4 Label geometry

When the beams are in perfect condition, the brush hits the bottle in its neutral position, $\theta = \theta_0$. Using the results of the previous section, we can calculate the change in gap size if the production is too fast and the bottle reaches the brush at $t < T$. At any time t, when the brush comes into contact with the bottle, assuming the beams remain straight, we have

$$y = x(t) = \sqrt{L^2 - (L + R - R\sin\theta - d_0)^2}$$

so that

$$\theta = \arcsin\left(\frac{L + R - d_0 - \sqrt{L^2 - x(t)^2}}{R}\right) .$$

The gap in the adhesion of the label is therefore larger by the amount $R(\theta - \theta_0)$. Figure 3 shows the change in gap size in meters as a function of t/T_r from 0 to 1. To generate this graph we chose $R = 5$cm, $L = 5$cm, $d_0 = 1$cm, $a = 4$cm, $n = 15000$ bottles per hour and $T = 0.05$ s. We can see that the change in gap size is relatively small when $t/T = 0.8$. This is true for any choice of T. The largest change in gap size is in the order of 1cm for our choice of R, L and d_0.

At the end of the process, the largest label that can be processed without rotation of the bottle has the length:

$$L = 2R\theta_f = 2R\left(\frac{\pi}{2} + \arccos\left(\frac{R - d_f}{R}\right)\right) . \tag{15}$$

The study of the process geometry leads to two major results:

1. An optimal production speed was derived that depends on the characteristics of the brushes and their geometry.

2. Studying the movement of the brushes indicated where they would apply pressure on the label.

These results insure that the brushes act as they are intended. Other aspects will now be considered.

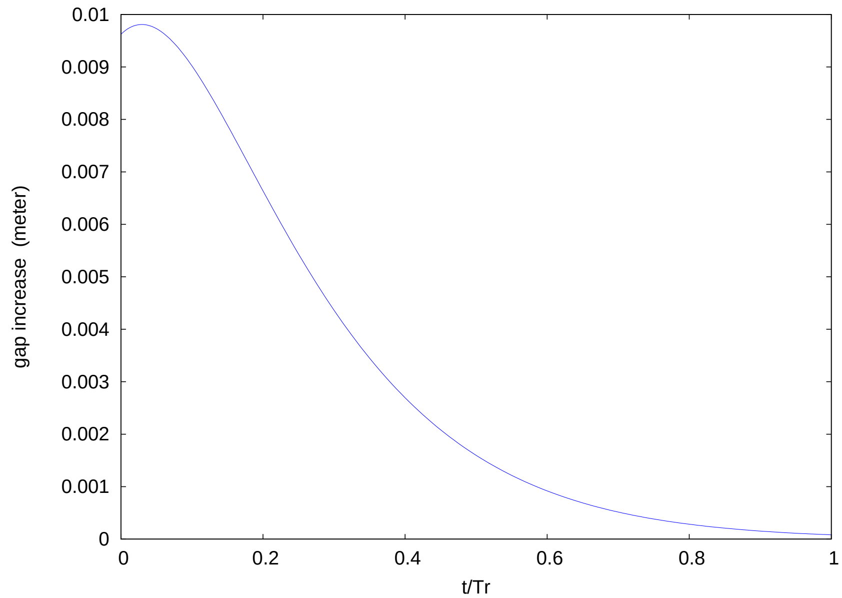


Figure 3: Change in gap size as a function of t/T_r.

4 Paper expansion

The use of the optimal production speed as studied in the previous section is one of several elements that may prevent the formation of wrinkles. The properties of paper will also play a key role. This will now be investigated and in this section, the distance between glue strips and the orientation of the paper label are determined.

4.1 Paper properties

Paper is an anisotropic material [5]. Its two directions are defined during the manufacturing process:

- The machine direction, denoted with the subscript 1 in the following. The majority of the fibres composing the paper are in this direction.
- The cross machine direction, orthogonal to the machine direction is denoted with the subscript 2 in the following.

The paper properties vary depending on the direction:

- The ratio of the elastic moduli is

$$\frac{E_1}{E_2} > 1 \ .$$

  The Young modulus is larger in the machine direction than in the cross direction. This ratio varies with the speed at which the paper is produced and may reach values up to 3.

- The Poisson ratio in the two directions are related to the Young moduli by the relation:

$$\frac{E_1}{E_2} = \frac{\nu_{12}}{\nu_{21}} \ .$$

- The shear modulus may be estimated as

$$G_{12} \approx \frac{1}{3}\sqrt{E_1 E_2} \ .$$

The various coefficients may be calculated as follows:

$$E_1 = \frac{\rho\Phi E^*}{16}\left(6 + 4a_1 + a_2\right)\left(1 - \nu_{12}\nu_{21}\right) \ , \tag{16}$$

$$E_2 = \frac{\rho\Phi E^*}{16}\left(6 - 4a_1 + a_2\right)\left(1 - \nu_{12}\nu_{21}\right) \ , \tag{17}$$

$$\nu_{12} = \frac{2 - a_2}{6 - 4a_1 + a_2} \ , \tag{18}$$

$$\nu_{21} = \frac{2 - a_2}{6 + 4a_1 + a_2} \ , \tag{19}$$

$$G_{12} = \frac{\rho\Phi E^*}{16}\left(2 - a_2\right) \ . \tag{20}$$

The constant Φ describes the effect of fibres, ρ and E^* are the paper density and specific elastic modulus respectively and

$$a_1 = 2q \ , \qquad a_2 = 2q^2 \ .$$

The parameter q is a constant varying with the speed, u_s, at which the paper is produced. A possible formula is

$$q = \tanh^2\left(\beta u_s\right) \ ,$$

where β is a constant.

When in humid conditions, the Young modulus values may be reduced by up to 30% and paper expands differently in the two directions. Typical values are around 0.4% in the cross machine direction and 0.12% in the machine direction [6]. The ratio between these two expansion coefficients is generally around 3. These values will now be used to model the formation of wrinkles.

4.2 Euler strut

A simple model for the wrinkling is provided by the Euler strut [7, 8]. A typical configuration may be found in Figure 4. The non-dimensional equation

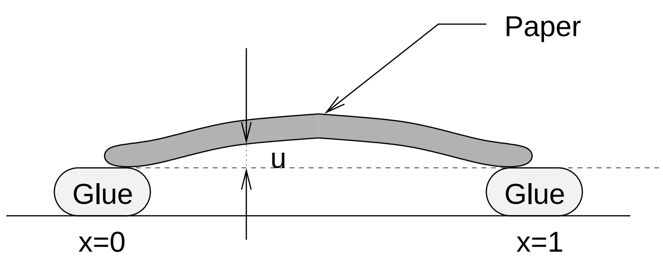


Figure 4: Typical configuration for the Euler strut.

governing u, the distance from the horizontal position, may be written:

$$\frac{d^2u}{dx^2} + \alpha^2 u = 0 \ , \tag{21}$$

where α is defined by

$$\alpha^2 = \frac{PL^2}{EI} \ ,$$

and P is the pressure due to paper expansion, E and I denote the Young modulus and the second moment of area of the paper respectively and L represents the length between two strips. Equation (21) may be solved analytically, subject to the boundary conditions:

$$u(0) = u(L) = 0 \ . \tag{22}$$

This means that the paper is in contact with the glue strips. This leads to:

$$u(x) = C \sin(\alpha x) \ ,$$

and the parameter α must satisfy the condition

$$\alpha L = n\pi \ , \quad n \in Z \ .$$

Physically, n corresponds to the number of oscillations between the two boundaries. In the present situation, the wrinkling corresponds to $n = 1$. This approach is consistent with the results presented in the Australia/New Zealand Study Group [1, 2]. According to this model, no wrinkling should occur for

$$L < \sqrt[4]{\frac{\pi^2 EI}{P}} \ . \tag{23}$$

In practice, wrinkling should be limited below this limit.

The value of the constant C may be calculated using the expansion of the paper. To evaluate this rate correctly, the scale for the height u must be the same as the length scale L. Figure 5 shows the expansion in the machine and cross machine directions for a similar pressure P, a ratio of Young moduli $E_1 = 2E_2$ and $\alpha_1 = 0.12\%$ and $\alpha_2 = 0.4\%$. The second moment varies with the cross-section and therefore is independent of the direction used. With these conditions, the ratio of length scales is

$$\frac{L_1}{L_2} = \left(\frac{E_1}{E_2}\right)^{1/4} = 2^{1/4} = 1.19 \ .$$

The two curves have approximately the same extent, which means that the direction of the paper does not significantly influence the formation of the wrinkles. The size of the wrinkles does however vary more significantly with the direction: the wrinkles are approximately twice as high in the cross machine direction, *i.e.* when the main fibre direction and the glue strips are parallel. The fibres and the glue strips should therefore preferably be orthogonal. The Euler strut method is a one-dimensional approach. A two-dimensional method will now be presented.

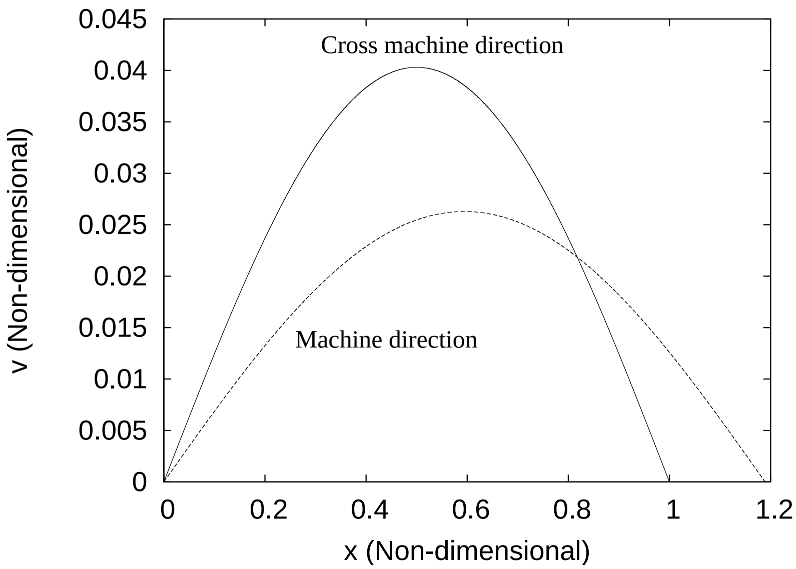


Figure 5: Paper expansion in machine and cross machine directions.

4.3 Two-dimensional paper deformation

Figure 6 shows the forces and momentum applied on a cross-section of paper [9]. If x_1 and x_2 denote the machine and cross directions of the paper respectively, at equilibrium, this leads to:

$$\frac{\partial T_2}{\partial x_1} + \frac{\partial T_1}{\partial x_2} + f = 0 \ , \tag{24}$$

$$\frac{\partial M_{11}}{\partial x_2} + \frac{\partial M_{12}}{\partial x_1} - T_1 = 0 \ , \tag{25}$$

$$\frac{\partial M_{22}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - T_2 = 0 \ , \tag{26}$$

where f denotes the force applied on the paper. Combining Equations (24–26) leads to:

$$\frac{\partial^2 M_{11}}{\partial {x_2}^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial {x_1}^2} + f = 0 \ .$$

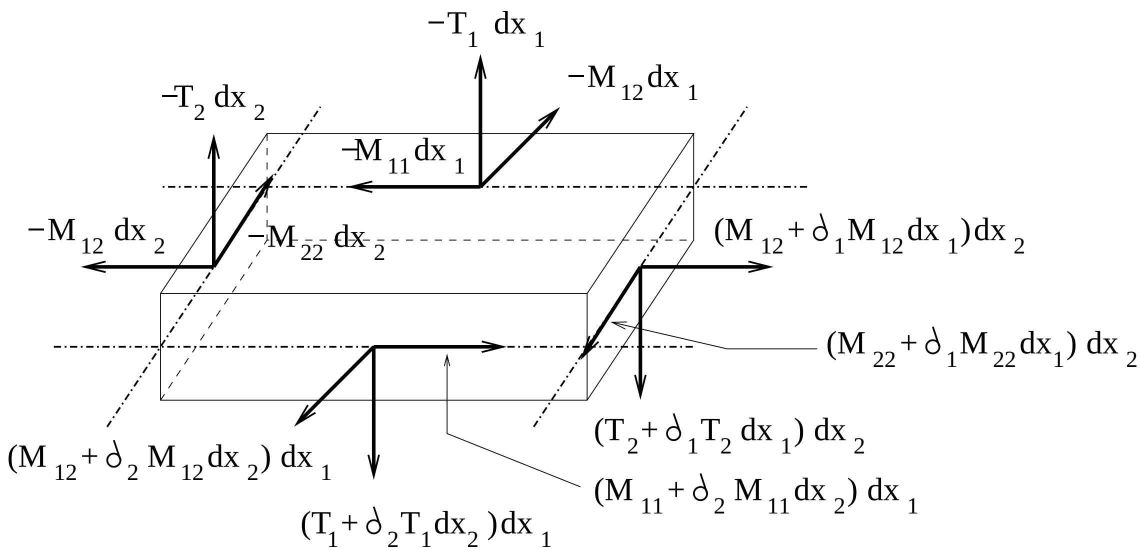


Figure 6: Forces across the paper.

The momentum values M_{11}, M_{12} and M_{22} may be expressed as:

$$M_{11} = \int_{-h/2}^{h/2} x_3\sigma_{22}dx_3\ ,\ M_{12} = \int_{-h/2}^{h/2} x_3\sigma_{12}dx_3\ ,\ M_{22} = \int_{-h/2}^{h/2} x_3\sigma_{11}dx_3\ , \tag{27}$$

where σ_{11}, σ_{12} and σ_{22} are the components of the Cauchy stress tensor. These expressions may be calculated using the relation

$$\begin{aligned}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix} &= \begin{pmatrix} 1/E_1 & -\nu_{12}/E_2 & 0\\ -\nu_{21}/E_1 & 1/E_2 & 0\\ 0 & 0 & 1/(2G_{12})\end{pmatrix}^{-1}\begin{pmatrix}e_{11}\\ e_{22}\\ e_{12}\end{pmatrix}\\ &= \begin{pmatrix} E_1E_2/(E_2-\nu_{21}^2E_1) & \nu_{21}E_1^2/(E_2-\nu_{21}^2E_1) & 0\\ \nu_{21}E_2^2/(E_2-\nu_{21}^2E_1) & E_2^2/(E_2-\nu_{21}^2E_1) & 0\\ 0 & 0 & 2G_{12}\end{pmatrix}\begin{pmatrix}e_{11}\\ e_{22}\\ e_{12}\end{pmatrix}\end{aligned} \tag{28}$$

where

$$e_{11} = -x_3\frac{\partial^2 u}{\partial {x_1}^2}\ ,\quad e_{22} = -x_3\frac{\partial^2 u}{\partial {x_2}^2}\ ,\quad e_{12} = -x_3\frac{\partial^2 u}{\partial x_1\partial x_2}\ , \tag{29}$$

and u is the vertical displacement. Combining Equations (27–29) leads to the following expressions:

$$M_{11} = -\frac{h^3}{12}\frac{E_2^2}{(E_2-\nu_{21}^2E_1)}\left(\nu_{21}\frac{\partial^2 u}{\partial {x_1}^2}+\frac{\partial^2 u}{\partial {x_2}^2}\right)\ , \tag{30}$$

$$M_{12} = -\frac{h^3}{6}G_{12}\frac{\partial^2 u}{\partial x_1\partial x_2}\ , \tag{31}$$

$$M_{22} = -\frac{h^3}{12}\frac{E_1}{(E_2-\nu_{21}^2E_1)}\left(E_2\frac{\partial^2 u}{\partial {x_1}^2}+\nu_{21}E_1\frac{\partial^2 u}{\partial {x_2}^2}\right)\ , \tag{32}$$

and the equation governing the vertical displacement may be written:

$$\frac{\partial^4 u}{\partial {x_1}^4}+\frac{E_2}{E_1}\frac{\partial^4 u}{\partial {x_2}^4}+\frac{4G_{12}(E_2-\nu_{21}^2E_1)+\nu_{21}(E_1^2+E_2^2)}{E_1^2}\frac{\partial^4 u}{\partial {x_1}^2\partial {x_2}^2} = \frac{12f(E_2-\nu_{21}^2E_1)}{h^3E_1^2}\ . \tag{33}$$

Equation (33) may be non-dimensionalised using the following scales:

$$u = Uu'\ , \qquad x_1 = X_1x_1'\ , \qquad x_2 = X_2x_2'\ .$$

Choosing

$$\frac{X_1}{X_2} = \left(\frac{E_2}{E_1}\right)^{1/4},\ \frac{U}{X_1^4} = \frac{12f(E_2-\nu_{21}^2E_1)}{h^3E_1^2}\ ,$$

$$\beta = \sqrt{\frac{E_1}{E_2}}\frac{4G_{12}(E_2-\nu_{21}^2E_1)+\nu_{21}(E_1^2+E_2^2)}{E_1^2}\ ,$$

and dropping the primes leads to the governing equation:

$$\frac{\partial^4 u}{\partial {x_1}^4}+\frac{\partial^4 u}{\partial {x_2}^4}+2\beta\frac{\partial^4 u}{\partial {x_1}^2\partial {x_2}^2} = 1\ . \tag{34}$$

The value of β may be calculated using Equations (16–20) and the corresponding results may be seen in Figure 7. For an isotropic paper, the ratio $E_1/E_2 = 1$ and in this situation, $\beta = 1$ and the governing equation (34) reduces to the standard bi-harmonic equation. The general solution for

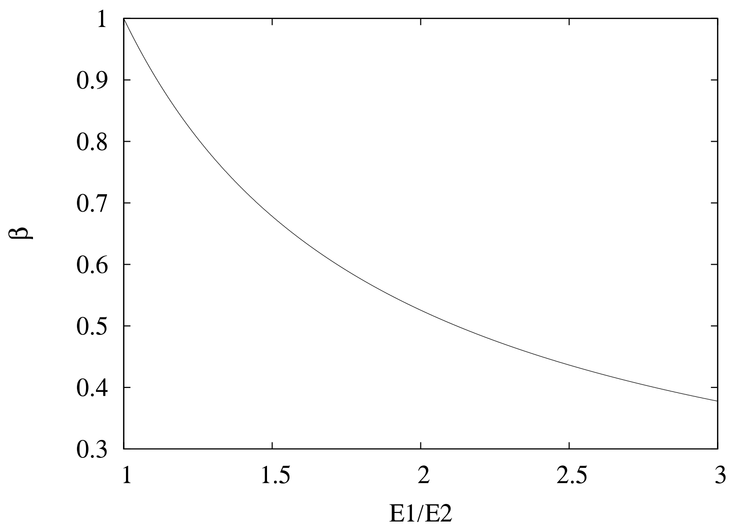


Figure 7: Values of β.

Equation (34) may be written:

$$\begin{aligned} u(x_1,\ x_2) = \frac{x_1^4}{24} \quad &+ \quad F_1\left(x_2 + x_1\sqrt{-\beta+\sqrt{\beta^2-1}}\right) \\ &+ \quad F_2\left(x_2 - x_1\sqrt{-\beta+\sqrt{\beta^2-1}}\right) \\ &+ \quad F_3\left(x_2 + x_1\sqrt{-\beta-\sqrt{\beta^2-1}}\right) \\ &+ \quad F_4\left(x_2 - x_1\sqrt{-\beta-\sqrt{\beta^2-1}}\right) \ . \end{aligned} \tag{35}$$

The unknown functions must be solved subject to the boundary conditions:

$$\begin{aligned} &u\left(x,\ x\tan(\theta)\right) = u\left(x,\ x\tan(\theta) + 1/\cos(\theta)\right) = 0 \ , \\ &\frac{\partial u}{\partial \mathbf{n}}\left(x,\ x\tan(\theta)\right) = \frac{\partial u}{\partial \mathbf{n}}\left(x,\ x\tan(\theta) + 1/\cos(\theta)\right) = 0 \ . \end{aligned}$$

Further analytical progress is difficult in the general case. Instead, numerical solutions could be computed using $E_1 = 2E_2$ and the corresponding value

$\beta \approx 0.52$. However, analytical solutions may be found when the fibres are in the x_1 and x_2 directions. In both cases, Equation (34) reduces to the ordinary differential equation:

$$\frac{d^4u}{dx^4} = 1 \ , \tag{36}$$

subject to the boundary conditions

$$u(0) = u(1) = u'(0) = u'(1) = 0 \ .$$

This leads to:

$$u = \frac{x^2\,(1-x)^2}{24} \ . \tag{37}$$

This curve must be rescaled to calculate paper expansion. The results may be seen on Figure 8. Here again, the cross machine direction leads to higher levels of wrinkling. The results are consistent with the outcome of the previous section. The higher rate of paper expansion in the cross machine direction drastically limits the benefits of the lower Young modulus. Putting the glue strips in the cross machine direction limits the height of the wrinkles but will not suppress them and they will appear for similar distances between the glue strips. Sufficiently reducing this distance should however limit the wrinkling effect significantly.

5 Spreading of a glue strip

The characteristic time for spreading of a glue strip is of primary importance in determining if wrinkles will form on the label. Sufficient time must be allowed when the label is attached for the width of the strip to spread on the surface of the bottle to the neighbouring glue strip.

The characteristic time for the spreading of a long glue strip of constant length was determined at the Australia/New Zealand MISG in 1996 [2] and later published [1]. We will first investigate the spreading of a glue strip of finite length which is not constant. We will then consider an expansion in terms of the ratio of the width to the length of the glue strip and re-derive the result established for a long glue strip of constant length as the zero order term in the expansion.

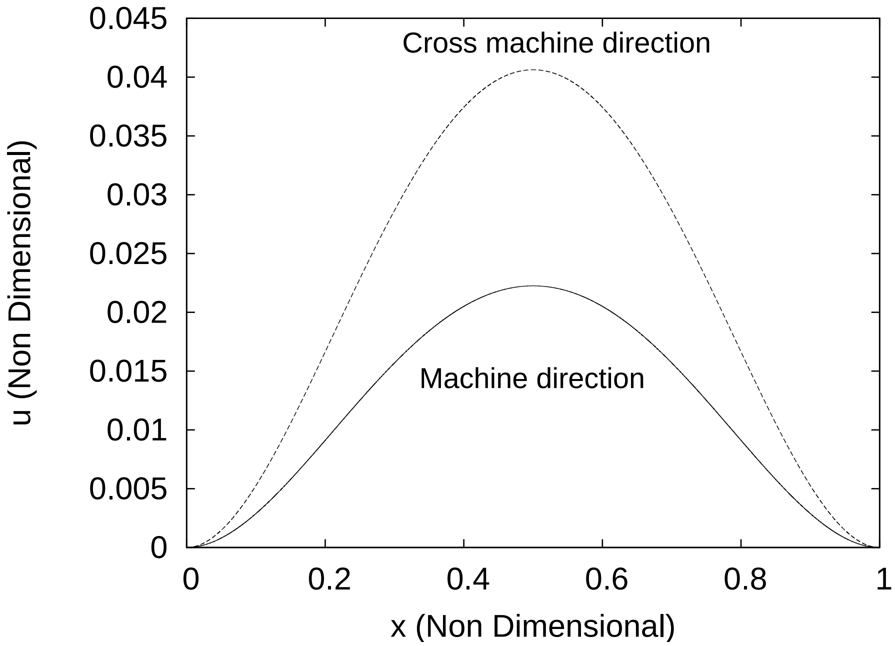


Figure 8: Paper expansion in machine and cross machine directions.

5.1 Governing equations

Consider a rectangular strip of glue that occupies the region

$$-L(t) \leq x \leq L(t)\ , \quad -W(t) \leq y \leq W(t)\ , \quad 0 \leq z \leq H(t)\ . \tag{38}$$

At time t the length of the glue strip is $2L(t)$, the width is $2W(t)$ and the height is $H(t)$. The origin of the coordinate system is at the centre of the base, $z = 0$. The top surface $z = H(t)$ is the label and the surface $z = 0$ is the surface of the bottle. Three equations governing $H(t)$, $L(t)$ and $W(t)$ are required to study the evolution of the glue strip. They may be derived from a mass balance, force balance and a third equation discussed in Section 5.3

5.1.1 Mass conservation

Since the glue is an incompressible fluid the total volume, V, of the glue strip remains constant. Thus $L(t)$, $W(t)$ and $H(t)$ satisfy the condition

$$L(t)W(t)H(t) = \frac{V}{4} \ . \tag{39}$$

5.1.2 Force balance

The force applied by the brushes on the label is balanced by the pressure applied by the glue. This pressure will now be calculated form the Navier-Stokes equations. The fluid velocity $\boldsymbol{v}$ and the fluid pressure p depend on x, y, z and time t:

$$v_x = v_x(x,y,z,t) \ , \ v_y = v_y(x,y,z,t) \ , \ v_z = v_z(x,y,z,t) \ , \ p = px,y,z,t) \ . \tag{40}$$

The glue strip is compressed by a force applied to the label. The label is assumed to be non-porous and therefore the velocity of the fluid normal to the label equals the velocity of the label:

$$z = H(t) : \qquad v_z(x,y,H(t),t) = \frac{dH}{dt} \ . \tag{41}$$

The normal component of the fluid velocity at the interface with the bottle vanishes:

$$z = 0 : \qquad v_z(x,y,0,t) = 0 \ . \tag{42}$$

Since the glue is attached to the bottle and to the label the horizontal fluid velocity is zero on both interfaces:

$$z = 0 : \qquad v_x(x,y,0,t) = 0 \ , \qquad v_y(x,y,0,t) = 0 \ , \tag{43}$$

$$z = H(t) : \qquad v_x(x,y,H(t),t) = 0 \ , \qquad v_y(x,y,H(t),t) = 0 \ . \tag{44}$$

The sides of the glue strip are open to the atmosphere. Atmospheric pressure is neglected so that

$$x = \pm L(t) : \qquad p\left(L(t),y,z,t\right) = 0 \ , \qquad p\left(-L(t),y,z,t\right) = 0 \ , \tag{45}$$

$$y = \pm W(t) : \qquad p\left(x,W(t),z,t\right) = 0 \ , \qquad p\left(x,-W(t),z,t\right) = 0 \ . \tag{46}$$

The lubrication approximation of the Navier-Stokes equations and the conservation of mass equation for an incompressible fluid are

$$\frac{\partial p}{\partial x} = \eta \frac{\partial^2 v_x}{\partial z^2} , \tag{47}$$

$$\frac{\partial p}{\partial y} = \eta \frac{\partial^2 v_y}{\partial z^2} , \tag{48}$$

$$\frac{\partial p}{\partial z} = 0 , \tag{49}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 , \tag{50}$$

where the body force due to gravity is neglected and η is the coefficient of dynamic viscosity. From (49),

$$p = p(x, y, t) . \tag{51}$$

In the lubrication approximation

$$\tau_{zz} (x, y, H(t), t) = -p(x, y, t) , \tag{52}$$

where τ_{zz} is the Cauchy stress tensor. Let F be the total force applied to the glue strip at the label. It must balance the fluid pressure in the glue strip acting on the label. Thus

$$F = \int_{-\mathrm{W}(t)}^{\mathrm{W}(t)} dy \int_{-L(t)}^{L(t)} p(x, y, t) dx . \tag{53}$$

In order to obtain F we have first to calculate $p(x, y, t)$.

Integrating (47) and (48) twice with respect to z and imposing the boundary conditions (43) and (44) at $z = 0$ and $z = H(t)$ gives

$$v_x(x, y, z, t) = \frac{1}{2\eta} z (z - H(t)) \frac{\partial p}{\partial x}(x, y, t) , \tag{54}$$

$$v_y(x, y, z, t) = \frac{1}{2\eta} z (z - H(t)) \frac{\partial p}{\partial y}(x, y, t) . \tag{55}$$

We next integrate the incompressibility condition (50) with respect to z from $z = 0$ to $z = H(t)$ and impose the boundary conditions (41) and (42). We find that

$$\frac{dH}{dt} = -\frac{\partial}{\partial x} \int_0^{H(t)} v_x(x, y, z, t) dz - \frac{\partial}{\partial y} \int_0^{H(t)} v_y(x, y, z, t) dx \tag{56}$$

and substituting (54) and (55) for v_x and v_y into (56) it can be verified that

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 12\eta \frac{1}{H^3(t)} \frac{dH}{dt} \ . \tag{57}$$

Poisson's equation (57) for $p(x, y, t)$ has to be solved subject to the boundary conditions (45) and (46).

We first make the change of variables from (x, y) to (r, s) where

$$r = \frac{1}{2}\left(x + L(t)\right) \ , \qquad s = \frac{1}{2}\left(y + W(t)\right) \ . \tag{58}$$

The glue strip is

$$0 \le r \le L(t) \ , \qquad 0 \le s \le W(t) \ , \qquad 0 \le z \le H(t) \ . \tag{59}$$

To determine the fluid pressure we employ an eigenfunction expansion of the form

$$p(r, s) = \sum_{n=1}^{\infty} b_n(s) \sin(\omega_n r) \ , \tag{60}$$

where the eigenvalue ω_n is

$$\omega_n = \frac{n\pi}{L(t)} \ . \tag{61}$$

The boundary condition (45) is identically satisfied. Substituting (60) into (57) gives

$$\sum_{n=1}^{\infty} \left(\frac{d^2 b_n}{ds^2} - \omega_n^2 b_n \right) \sin\left(\omega_n r\right) = 48\eta \frac{1}{H^3(t)} \frac{dH}{dt} \ . \tag{62}$$

By multiplying both sides of (62) by $\sin\left(\omega_m r\right)$, integrating with respect to r from $r = 0$ to $r = L(t)$ and using the identities

$$\int_0^{L(t)} \sin\left(\omega_n r\right) \sin\left(\omega_m r\right) dr = \frac{1}{2} L(t) \delta_{nm} \ , \tag{63}$$

$$\int_0^{L(t)} \sin\left(\omega_m r\right) dr = \frac{1}{\omega_m} \left(1 - (-1)^m\right) \ , \tag{64}$$

it can be shown that

$$\frac{d^2 b_n}{ds^2} - \omega_n^2 b_n = q_n(t) \ , \qquad 1 \le n \le \infty \ , \tag{65}$$

where

$$q_n(t) = 96\eta \frac{1}{H^3(t)} \frac{dH}{dt} \frac{(1-(-1)^n)}{n\pi} \ . \tag{66}$$

The general solution of (65) is

$$b_n(s) = A_n(t)\cosh(\omega_n s) + B_n(t)\sinh(\omega_n s) - \frac{q_n(t)}{\omega_n^2} \ . \tag{67}$$

From (46) and the eigenfunction expansion (60), the boundary conditions on $b_n(s)$ are

$$b_n(0) = 0 \ , \qquad b_n(\mathrm{W}) = 0 \ . \tag{68}$$

Thus

$$b_n(s) = \frac{q_n(t)}{\omega_n^2}\left[\cosh(\omega_n s) - \left(\frac{\cosh(\omega_n \mathrm{W}) - 1}{\sinh(\omega_n \mathrm{W})}\right)\sinh(\omega_n s) - 1\right] \ . \tag{69}$$

From (53) and (58) the total force applied to the glue strip is

$$F = 4\int_0^{W(t)} ds \int_0^{L(t)} p(r,s)dr \ . \tag{70}$$

Using (69) it is readily verified that

$$\begin{aligned} F &= 4\sum_{n=1}^{\infty} \frac{q_n(t)}{\omega_n^4}\left[\sinh(\omega_n W) - \frac{(\cosh(\omega_n W) - 1)^2}{\sinh(\omega_n W)} - \omega_n W\right][1-(-1)^n] \\ &= 4\sum_{n=1}^{\infty} \frac{q_n(t)}{\omega_n^4}\left[\frac{2(\cosh(\omega_n W) - 1)}{\sinh(\omega_n W)} - \omega_n W\right][1-(-1)^n] \ . \end{aligned} \tag{71}$$

Substitution of (66) for $q_n(t)$ in (71) gives

$$F = \frac{384\,\eta}{L(t)H^3(t)} \frac{dH}{dt} \sum_{n=1}^{\infty} \frac{1}{\omega_n^5}\left[\frac{2(\cosh(\omega_n W) - 1)}{\sinh(\omega_n W)} - \omega_n W\right][1-(-1)^n]^2 \ . \tag{72}$$

5.2 Expansion of the force balance for small values of W/L.

Now

$$\omega_n W = n\pi \frac{W}{L} \ . \tag{73}$$

The length of a glue strip is about 10 cm and its width is about 1mm. Hence

$$\frac{W}{L} = O\left(10^{-2}\right) \ . \tag{74}$$

We can therefore consider an expansion of (72) for small values of $\omega_n W$ although we can expect mathematical difficulties from the contribution by eigenfunctions with large values of n.

Since

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \ , \qquad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \ , \tag{75}$$

it can be shown that

$$\begin{aligned} F \ = \ 384\,\eta \frac{L^4(t)}{H^3(t)} \frac{dH}{dt} \Bigg[& -\frac{1}{12\pi^2}\left(\frac{W}{L}\right)^3 \sum_{n=1}^{\infty} \frac{\left(1-(-1)^n\right)^2}{n^2} \\ & + \frac{1}{120}\left(\frac{W}{L}\right)^5 \sum_{n=1}^{\infty} \left(1-(-1)^n\right)^2 \\ & + O\left(\left(\frac{W}{L}\right)^7\right) \sum_{n=1}^{\infty} n^2 \left(1-(-1)^n\right)^2 \Bigg] \ . \end{aligned} \tag{76}$$

Now

$$\sum_{n=1}^{\infty} \frac{\left(1-(-1)^n\right)^2}{n^2} = 4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \ . \tag{77}$$

But [10]

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\zeta(2) = \frac{\pi^2}{8} \ , \tag{78}$$

where the Riemann Zeta function $\zeta(2)$ satisfies

$$\zeta(2) = \frac{\pi^2}{6} \ . \tag{79}$$

Thus

$$\sum_{n=1}^{\infty} \frac{\left(1-(-1)^n\right)^2}{n^2} = \frac{\pi^2}{2} \ . \tag{80}$$

The other two summations in (76) are divergent. Equation (75) becomes

$$F = -16\eta L(t) \frac{W^3(t)}{H^3(t)} \frac{dH}{dt} \left[1 + O\left(\left(\frac{W}{L}\right)^2\right) \sum_{n=1}^{\infty} \left(1-(-1)^n\right)^2\right] \ . \tag{81}$$

Although $\left(\frac{W}{L}\right)^2 = O\left(10^{-4}\right)$ the expansion breaks down after the zero order term. The zero order term in the expansion (81) gives

$$F = -16\eta L(t) \frac{W^3(t)}{H^3(t)} \frac{dH}{dt} \ , \tag{82}$$

in agreement with the result derived assuming that $L(t)$ is large and constant [1, 2]. The applied force F is positive. The negative sign is because $H(t)$ is a decreasing function of t.

By considering an eigenfunction expansion in the x-direction we were able to derive a series expansion in powers of W/L. If the eigenfunction expansion had been made in the y-direction the terms would depend on L/W which would not yield a series expansion when $L \gg W$.

Two equations, (39) and (82), have now been derived for $L(t)$, $W(t)$ and $H(t)$. One further condition is required.

5.3 Third equation and solutions

5.3.1 Constant length approximation

Consider first the approximation that $L(t)$ remains constant. Let $L(0) = L_0$. Then from (39),

$$W(t) = \frac{A}{2H(t)} \ , \tag{83}$$

where $A = V/2L_0$ is the constant cross-sectional area of the glue strip. Equation (82) for $H(t)$ becomes

$$\frac{dH}{dt} = -\frac{F}{2\eta L_0 A^3} H^3 \ . \tag{84}$$

Integration of (84) subject to the initial condition $H(0) = H_0$ gives

$$H(t) = \frac{H_0}{\left(1 + \dfrac{t}{t_s}\right)^{\frac{1}{5}}} \ , \tag{85}$$

where

$$t_s = \frac{2\eta L_0 A^3}{5FH_0^5} \ , \tag{86}$$

in agreement with existing results [1, 2]. The time t_s is the characteristic time for the glue to spread.

In order to rewrite (86) in a more practical form we express it in terms of the mean pressure P applied to the label [1, 2]. The length of the label is $2L_0$. If there are N glue strips and the breadth of the label is $2B_0$ then

$$P = \frac{NF}{4L_0B_0} \ . \tag{87}$$

Expressed in terms of P, (77) becomes

$$t_s = \frac{\eta N A^3}{10 B_0 P H_0^5} \ , \tag{88}$$

where $A = 2W_0H_0$ is the constant cross-sectional area of the glue strip.

5.3.2 Infinite series solution.

We return to (72) which may be written in the form

$$\frac{dH}{dt} = -G(t)H^3(t), \tag{89}$$

where

$$G(t) = -\frac{FL(t)}{384\,\eta \sum_{n=1}^{\infty} \frac{1}{w_n^5}\left[\frac{2\left(\cosh\left(\omega_n W\right)-1\right)}{\sinh\left(\omega_n W\right)} - \omega_n W\right]^2 \left[1-(-1)^n\right]^2} \ . \tag{90}$$

We approximate $G(t)$ by its value, G_0, at $t = 0$ obtained by setting $W(t) = W(0) = W_0$ and $L(t) = L(0) = L_0$. Then if $H = H_0$ at $t = 0$,

$$H(t) = \frac{H_0}{\left(1 + \frac{t}{t_s}\right)^{\frac{1}{2}}} \tag{91}$$

where

$$t_s = \frac{1}{2G_0H_0^2} \ . \tag{92}$$

To compare (92) with (86) consider the expansion of G_0 to lowest order in W/L. From (81), and to lowest order in W/L,

$$G_0 = \frac{F}{16\eta L_0 W_0^3} \tag{93}$$

and therefore to lowest order in W/L, (92) gives

$$t_s = \frac{8\eta L_0 W_0^3}{F H_0^2} = \frac{\eta L_0 A^3}{F H_0^5}. \tag{94}$$

Combining the three governing equations, the width of the strip $W(t)$ can be evaluated, leading directly to the distance between two consecutive strips. To limit wrinkling, this distance should be lower than the value defined by formula (23). This occurs at time t_0. Since each brush only applies pressure on the label for the time t_f, defined by formula (14), the number of brushes necessary to label the bottles and limit wrinkles may be estimated as t_0/t_f.

6 Removal of the label

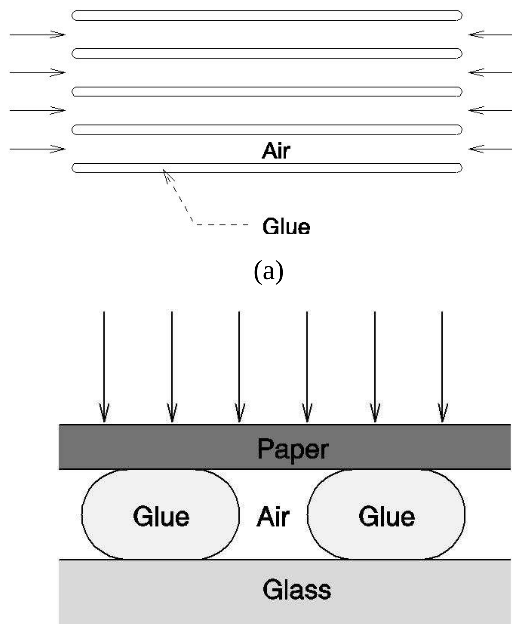
The removal of the labels is now briefly considered. Reducing the space between the glue strips may prevent the labels from being removed easily. In this final stage of the labelling, two processes compete:

- Water dissolves the glue from the sides of the label, as shown on Figure 9a. The water moves by capillarity in the space located between two consecutive glue strips, the label and the bottle.
- Water diffuses through the label and dissolves the glue from the label side, see Figure 9b.

Simple experiments were conducted during the Study Group and they show that the second process is dominant: the label is unglued everywhere at approximately the same time and there is a significant amount of glue left on the bottle. (According to the SAB representative, the penetration of the label happens 90% through the label and 10% penetration from the edges.) This shows that reducing the space between glue strips as suggested in previous sections should not affect the removal of the labels significantly.

7 Conclusion

This Study Group focused on some key aspects of this wide ranging problem:



(a) (b)

Figure 9: Processes in competition when removing of the labels.

- **Optimal processing speed and consequences on the label geometry**
  An optimal processing speed was calculated: it depends on the geometry of the bottle and the physical properties of the brushes. Using this optimal speed maximises the proportion of the label that can be reached by the brushes. The geometry also gives the largest label size that can be accommodated by the current set-up. The gaps could further be reduced and the label size extended by turning the bottles slightly towards either side when passing through the brushes. This study also provides the amount of time the brushes are in contact with the label.

- **Geometry of the glue strips**
  The geometry of the glue strips is a key element in the wrinkling effect. The group evaluated the importance of the distance between the glue strips. A maximum distance was determined between strips after

spreading: below this value, the wrinkling should be significantly reduced. The group also investigated the influence of the angle between the preferential expansion direction of the paper and the glue strips. Glue strips should be perpendicular to the fibres to reinforce the paper against expansion.

- **Spreading of the glue**
  The evolution of a glue strip was studied when pressure is applied. Two models were developed. They provide the minimum amount of time pressure should be applied on the label to reach the spacing between strips that would prevent wrinkling. Coupled with the results of the geometry study, this gives the number of brushes necessary to label the bottles and limit the wrinkling.

- **Label removal**
  The removal of labels was also briefly considered. The group studied the different processes leading to this removal and assessed the consequences of changing the glue patterns. Water diffusion through the label is the leading process. A modification of the glue distribution should therefore not affect the label removal drastically.

In future work, the models used in this study should be further developed and validated using experimental results.

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