A Six Degree of Freedom Model for a Submersible

1. Introduction

The primary aim of the study group was to identify alternative approaches or improvements to the existing method of predicting submarine motion. A secondary problem of interest was that of roll instability when a submarine has recently surfaced. This latter situation results from a large amount of water under the casing which takes some considerable time to drain away.

One option currently being considered in the MOD for the main problem of submarine motion is a vortex lift-line model. It was generally thought that this avenue would be worth pursuing but due to security restrictions, little information was available and so no work was done on this system.

The present model uses the general equations of rigid body motion resolved along axes fixed in the submarine with the external forces and moments left as unknown functions (see Appendix). The 6 degrees of freedom are then the 3 linear plus 3 angular velocity components. The general equations are linearized by considering small perturbations about steady forward motion and the partial derivatives for the unknown forces and moments are evaluated using experimental and computational fluid dynamics techniques.

Our main efforts were concentrated on suggesting improvements to the current linear model which is inaccurate when the submarine undergoes rapid changes in direction. Then neglected nonlinear terms become important and an *ad hoc* system for adding these in has been developed. A more rigorous approach requires a return to the general equations and a better understanding of the fluid dynamical effects and their resulting forces and moments acting on the submarine. Various types of fluid effects were discussed, including buoyancy, fluid inertia ("added mass"), viscous drag on the hull due to skin friction, lift forces on the rudder, hydroplanes, and conning tower and viscous and history effects due to vortex shedding, and where possible analytical expressions were given.

Once a more complete model for the submarine plus the fluid has been obtained, it may then be possible to simplify it using perturbation methods, but it will almost certainly require numerical techniques to be of practical use.

2. Submersible Coordinates

There appears to be some inconsistency in the notation used as in the list of control notation provided by the MOD, x_G, y_G, z_G are given as the body axes with I_x, I_y and I_z the moments of inertia about these axes, whereas in the general equations of motion x_G, y_G, z_G are the coordinates of the centre of gravity. We also note that the products of inertia are the negative of the normal definitions. The general equations of motion are derived in the Appendix. We shall use the following notation (see Figures 1a, 1b):-

x, y, z	body axes
I_x, I_y, I_z	moments of inertia about x, y, z-axes
I_{xy}, I_{yz}, I_{zx}	products of inertia about x, y, z-axes
ϕ, θ, ψ	roll, pitch, yaw angles (Figure 1b)
u, v, w	linear velocity components
p, q, r	angular velocities $(\dot{\phi}, \dot{\theta}, \dot{\psi})$
X, Y, Z	force components
K, M, N	moment components

ρ	density of water
m	mass of submarine
V	volume of submarine
L	length of submarine
G	centre of gravity
B	centre of buoyancy
x_g, y_g, z_g	coordinates of G
x_b, y_b, z_b	coordinates of B
$\delta B, \delta S, \delta R$	bow plane, stern plane, rudder deflection angles
$\mathbf{F}$	force vector (X, Y, Z)
$\mathbf{N}$	moment vector (K, M, N)
$\mathbf{\Omega}$	angular velocity vector (p, q, r)
$\mathbf{g}$	gravitational force.

Other symbols are defined where they arise in the text.

3. Gravitational and Buoyancy Effects

The total force on the submarine from the gravitational and buoyancy effects is given by

$$\mathbf{F} = m\mathbf{g} - \rho V \mathbf{g}, \tag{3.1}$$

where m is the mass of the submarine, ρ is the density of water and V is the volume of water displaced (the volume of the submarine). If we define $\mathbf{r}_g$ to be the position of the centre of gravity and $\mathbf{r}_b$ to be the position of the centre of buoyancy in sub–coordinates, then the moment $\mathbf{N}$ about the origin is given by

$$\mathbf{N} = m\mathbf{r}_g \times \mathbf{g} - \rho V \mathbf{r}_b \times \mathbf{g}. \tag{3.2}$$

A problem arises when trying to resolve $\mathbf{g}$ in the sub-coordinates for a general position of the submarine using the angles ϕ, θ, ψ. We note that there is not a unique decomposition since a roll of 90° followed by a pitch of angle α is equivalent to a yaw of angle α followed by a roll of 90° so that the order of the rotations is important. (In fact only two angles are necessary to define the position, cf. spherical polar coordinates.)

A transformation can be written down for each of the pitch, roll and yaw motions respectively

$$\mathbf{x} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \mathbf{x}', \tag{3.3}$$

$$\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \mathbf{x}', \tag{3.4}$$

$$\mathbf{x} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}', \tag{3.5}$$

where $\mathbf{x}$ is in the sub-frame and $\mathbf{x}'$ in the rotated axes. Thus if z' is vertically downwards so that $\mathbf{g} = (0, 0, g)$ in the primed axes, then it is given by

$$\mathbf{g} = \begin{pmatrix} 0 \\ g \sin \phi \\ g \cos \phi \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -g \sin \theta \\ 0 \\ g \cos \theta \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}, \tag{3.6}$$

in the sub-axes for the 3 cases. As matrix multiplication is not commutative, the order of the rotations matters as already stated. However, if we linearize the rotation matrices by replacing the cos terms by 1 and the sin terms by their argument, we obtain near identity matrices. Then neglecting the product of angles, the order of multiplication does not matter and we obtain the composite transformation

$$\mathbf{x} = \begin{pmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{pmatrix} \mathbf{x}'. \tag{3.7}$$

Thus again taking $\mathbf{g} = (0, 0, g)$ in the primed axes, we can approximate $\mathbf{g}$ by

$$\mathbf{g} = \begin{pmatrix} -\theta \\ \phi \\ 1 \end{pmatrix} g \tag{3.8}$$

in the sub-frame of reference.

For a neutrally buoyant submarine,

$$m = \rho V, \tag{3.9}$$

and so there is no linear force from (3.1), but there is still a turning moment given by (3.2). We assume that the y-components of both $\mathbf{r}_g$ and $\mathbf{r}_b$ can be neglected so we can write

$$\mathbf{r}_g = \begin{pmatrix} x_g \\ 0 \\ z_g \end{pmatrix}, \quad \mathbf{r}_b = \begin{pmatrix} x_b \\ 0 \\ z_b \end{pmatrix}. \tag{3.10}$$

Now eliminating ρV from (3.2) using the neutral buoyancy condition (3.9), we find

$$\mathbf{N} = m(\mathbf{r}_g - \mathbf{r}_b) \times \mathbf{g}. \tag{3.11}$$

Then a substitution for $\mathbf{r}_g$, $\mathbf{r}_b$ and $\mathbf{g}$ from (3.8) and (3.10) shows that the moment of the gravitational and buoyancy forces can be approximated for small roll and pitch angles by

$$\mathbf{N} = -mg(z_g - z_b) \begin{pmatrix} \phi \\ \theta \\ 0 \end{pmatrix} - mg(x_g - x_b) \begin{pmatrix} 0 \\ 1 \\ -\phi \end{pmatrix}. \tag{3.12}$$

We note that the submarine will not maintain level uniform motion unless

$$x_g = x_b, \tag{3.13}$$

as otherwise there is a large pitching moment. We require a stable state to linearize about so we assume that equation (3.13) holds and we are then left with

$$\mathbf{N} = -mg(z_g - z_b) \begin{pmatrix} \phi \\ \theta \\ 0 \end{pmatrix}. \tag{3.14}$$

This agrees with the terms given in the linearized equations as $z_g - z_b = BG_v$ (i.e. the vertical separation of B and G).

4. Added Mass and Inertia

When a body is accelerating in fluid which was initially at rest, added mass and inertia terms arise. For a body translating with speed $U(t)$ through an irrotational liquid with no circulation around the body then

$$DU = \frac{dT}{dt}, \tag{4.1}$$

where D is the drag and $T(t)$ is the total kinetic energy of the fluid. This equation can be used to calculate the drag, but it breaks down if there is separation of the flow and in a viscous fluid where energy is dissipated. For a sphere the drag is $\frac{1}{2}\rho V \frac{dU}{dt}$, where ρV is the mass of fluid displaced and for a cylinder moving perpendicular to its axis, $D = \rho V \frac{dU}{dt}$. Therefore, we have a relation of the form

$$\mathbf{F} = -M \left(\frac{d\mathbf{U}}{dt}\right)_{in}, \tag{4.2}$$

where M is a mass tensor of second order and the time derivative of $\mathbf{U}$ is the acceleration in an inertial frame (Batchelor 1967 p.407). For a rotating body,

$$\left(\frac{d\mathbf{U}}{dt}\right)_{in} = \dot{\mathbf{U}} + \boldsymbol{\Omega} \times \mathbf{U}, \tag{4.3}$$

where '$\cdot$' is the time derivative in sub-coordinates and $\boldsymbol{\Omega}$ is the angular velocity of the submarine (see Appendix).

Assuming that equation (4.2) can still be applied with small rotation, and that the main components of the mass tensor are on the diagonal then we have

$$M \approx \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \tag{4.4}$$

and so with rotation,

$$\mathbf{F} = -\begin{pmatrix} m_1(\dot{u} + qw - rv) \\ m_2(\dot{v} + ru - pw) \\ m_3(\dot{w} + pv - uq) \end{pmatrix}. \tag{4.5}$$

We can make the further approximation of the velocity of the submarine being primarily in the forward direction, and set

$$u = U_0 + u', \qquad v = v', \qquad w = w', \quad p = p', \quad q = q', \quad r = r' \tag{4.6}$$

where the primes denote small quantities. Then just retaining the linear terms in the perturbations (and dropping the primes), the drag term can be simplified to

$$\mathbf{F} = -\begin{pmatrix} m_1\dot{u} \\ m_2(\dot{v} + rU_0) \\ m_3(\dot{w} - U_0 q) \end{pmatrix}. \tag{4.7}$$

Similarly, for the moments we can write

$$\mathbf{N} = -\frac{d}{dt}(I\boldsymbol{\Omega}), \tag{4.8}$$

where I is an inertia tensor and this could be approximated in the first instance by

$$\mathbf{N} = -\begin{pmatrix} I_1\dot{p} \\ I_2\dot{q} \\ I_3\dot{r} \end{pmatrix} - \begin{pmatrix} \dot{I}_1 p \\ \dot{I}_2 q \\ \dot{I}_3 r \end{pmatrix}, \tag{4.9}$$

where I_1, I_2, I_3 are the added moments of inertia about the sub-axes due to the fluid motion.

5. Viscous Drag

In a high Reynolds number flow produced by a body moving steadily through fluid at rest at infinity, there is irrotational flow outside the boundary layer and wake if no separation occurs. Then the total friction drag on the body in translational motion is given by (Batchelor 1967 p.335)

$$D = k\rho U^2 a R^{-\frac{1}{2}}, \tag{5.1}$$

where U is the velocity, a is a measure of the body surface, k is a number depending on the body shape and $R = UL/\nu$ is the Reynolds number with L an appropriate length-scale and ν the viscosity of water. Thus, we expect the drag to take the form

$$\mathbf{F} = -A\mathbf{U}|\mathbf{U}|^{\frac{1}{2}}, \tag{5.2}$$

where A is a second order tensor and for small departures from the state of uniform translation using (4.6), we can approximate this by

$$\mathbf{F} = -\begin{pmatrix} k_1(U_0^{\frac{3}{2}} + \frac{3}{2}U_0^{\frac{1}{2}}u') \\ k_2 U_0^{\frac{1}{2}} v' \\ k_3 U_0^{\frac{1}{2}} w' \end{pmatrix}. \tag{5.3}$$

The largest term in the drag force is balanced by the engine thrust which acts in the positive x-direction.

6. Forces on the Rudder

The rudder acts as an aerofoil and provided its angle of attack with the flow is not too large, then we can use classical aerofoil theory. Therefore, we have no drag force in the direction of the flow, but a lift force exists perpendicular to the velocity. We suppose that the submarine is translating with velocity U_0 in the x-direction and is not rotating. Then the effect of turning the rudder through an angle δR is to produce a lift force of magnitude $-\rho U_0 \Gamma$ in the y-direction, where Γ is the circulation around the aerofoil. For a thin plate of length l or a narrow Joukowski aerofoil, the circulation is given by

$$\Gamma = -4\pi U_0 l \sin \delta R, \tag{6.1}$$

per unit length of aerofoil for flow at an angle δR (Acheson 1990, p.121). Thus the total instantaneous lift produced is $4\pi\rho A_R U_0^2 \sin \delta R$ in the y-direction, where A_R is the surface area of the rudder. For small deflections, $\sin \delta R$ can be approximated by δR to give

$$\mathbf{F} = \begin{pmatrix} 0 \\ 4\pi\rho A_R U_0^2 \delta R \\ 0 \end{pmatrix}. \tag{6.2}$$

The upper and lower rudders are of different size and we define ΔA_R to be the difference in their surface areas. Thus the centre of lift has a significant z-coordinate, z_R, as well an x-coordinate, x_R. This leads to a turning moment of

$$\mathbf{N} = 4\pi\rho U_0^2 \delta R \begin{pmatrix} \Delta A_R z_R \\ 0 \\ -A_R x_R \end{pmatrix}. \tag{6.3}$$

The component in the z-direction leads to yaw, whilst the smaller term gives rise to roll.

7. Forces on the Hydroplanes

In the same way as for the rudder, we can calculate the instantaneous forces on the hydroplanes when they are moved to angles δP during uniform motion of the submarine. The lift force is in the $-z$-direction and has magnitude $4\pi\rho U_0^2 A_P \delta P$, where A_P is the area of the hydroplane. There are 4 of these surfaces but they are connected in pairs with the same surface area and angle to the flow (δB for the bow planes and δS for the stern planes). Thus any roll moment cancels and the only contribution will be to pitch, so

$$\mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ -\sum 4\pi\rho U_0^2 A_P \delta P \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 \\ \sum x_P 4\pi\rho U_0^2 A_P \delta P \\ 0 \end{pmatrix}, \tag{7.1}$$

where we sum over the 4 hydroplanes and x_P is the x-coordinate of the plane. Again, these results are only valid for small angles of attack as if the angle is too large then the boundary layer separates, giving rise to a turbulent wake and a sudden drop in lift as the aerofoil stalls.

8. Lift Forces on the Submersible Hull

Lift forces on the conning tower and submarine as a whole are only important during manoeuvres when the velocity is at an angle to the body. For simplicity, we consider what happens when the submarine is undergoing translational motion only and separate the behaviour into symmetric and antisymmetric motion, as done in the linear model.

For symmetric motion, there is no velocity component in the y-direction and no roll or yaw. Thus we have the linear velocity

$$\mathbf{U} = \begin{pmatrix} U_0 + u' \\ 0 \\ w' \end{pmatrix}, \tag{8.1}$$

for small perturbations from uniform forward motion. The main component of lift is $-\rho\Gamma U_0$ in the $-z$-direction if there is a circulation Γ. Even if there is no circulation, we can still have a turning moment and for an elliptical cylinder, this can be approximated by

$$\mathbf{N} = \begin{pmatrix} 0 \\ 4\pi\rho U_0 L^2 w' \\ 0 \end{pmatrix}, \tag{8.2}$$

where L is the submarine length (Acheson 1990 p.143).

For antisymmetric motion, we have no velocity component in the z-direction and no pitch. Now the velocity is given by

$$\mathbf{U} = \begin{pmatrix} U_0 + u' \\ v' \\ 0 \end{pmatrix}, \tag{8.3}$$

and the main component of lift is $-\rho\Gamma U_0$ in the $-y$-direction for a circulation Γ. The turning moment this time contributes to yaw and can be written as

$$\mathbf{N} = \begin{pmatrix} 0 \\ 0 \\ -4\pi\rho U_0 L^2 v' \end{pmatrix}. \tag{8.4}$$

In the antisymmetric case, we also have an additional contribution due to the conning tower, which acts as an aerofoil. Then to leading order, the lift force is

$$\mathbf{F} = 4\pi\rho U_0 A_c \begin{pmatrix} 0 \\ -v' \\ 0 \end{pmatrix}, \tag{8.5}$$

and has moment

$$\mathbf{N} = -4\pi\rho U_0 A_c \begin{pmatrix} v' L_{cz} \\ 0 \\ -v' L_{cx} \end{pmatrix}, \tag{8.6}$$

where L_{cx} and L_{cz} are the x and z components respectively of the centre of lift on the conning tower and A_c is the area. If in addition, we have small rotation, then defining the position vector of the centre of the conning tower by

$$\mathbf{r}_c = \begin{pmatrix} x_c \\ 0 \\ z_c \end{pmatrix}, \tag{8.7}$$

the velocity of this point is

$$\mathbf{V} = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}_c. \tag{8.8}$$

In this case, the angular velocity is given by

$$\boldsymbol{\Omega} = \begin{pmatrix} p \\ 0 \\ r \end{pmatrix},$$

and so in components, we have

$$\mathbf{V} = \begin{pmatrix} U_0 + u' \\ v' - pz_c + rx_c \\ 0 \end{pmatrix}. \tag{8.9}$$

Thus we can replace v' in (8.5) and (8.6) by $v' - pz_c + rx_c$ to obtain the appropriate force and moment for small rotation.

9. Vortex Shedding

Vortex shedding becomes a problem when either the angle of attack of one of the hydroplanes is too high or when the submarine is undergoing rapid manoeuvring. This leads to additional drag forces and it was thought possible that history effects might be important if part of the submarine moved through a previously shed vortex. However, this is unlikely to occur as we will see in this section.

If $\mathbf{r} = (x, y, z)$ is the position vector of a point on the submarine hull, then the velocity of this point is given by

$$\mathbf{V} = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r} = \begin{pmatrix} U_0 + u' + qz - ry \\ v' - pz + rx \\ w' + py - qx \end{pmatrix}, \tag{9.1}$$

for arbitrary rotation of the submarine. If we restrict ourselves to considering a turn in a horizontal circle, then $p \ll r$, $q \ll r$ and the front and back of the submarine are given by $(\pm L/2, 0, 0)$ so that equation (9.1) simplifies to

$$\mathbf{V} = \begin{pmatrix} U_0 + u' \\ v' \pm rL/2 \\ 0 \end{pmatrix}. \tag{9.2}$$

The angle β between the front or back of the submarine and its local velocity is given by

$$\tan\beta = \frac{v' \pm rL/2}{U_0 + u'}. \tag{9.3}$$

If r is positive so that the yaw angle is increasing in the turn, then v' will be negative if the submarine is side-slipping. Thus there is a larger angle at the back and so vortex shedding is more likely there. In this situation, history effects will be unimportant as the vortices will be left behind.

Defining the diameter of the turning circle to be T_D, the angular velocity r can be approximated by

$$r = \frac{2U_0}{T_D}, \tag{9.4}$$

and so

$$\beta \approx \frac{v'}{U_0} - \frac{L}{T_D}, \tag{9.5}$$

at the rear of the submarine. There will then be a bluff-body drag contribution in the X and Y directions proportional to

$$\left(\frac{v'}{U_0} - \frac{L}{T_D}\right)^2,$$

although we note that the linearization is no longer appropriate if T_D is too small.

10. Conclusions and Further Work

The linear method currently used is virtually identical to the aircraft stability derivatives model (Duncan 1952, Duncan *et al.* 1970). It would be worthwhile conducting a thorough survey of the aircraft literature to see what has been done as the problems are similar. For military purposes, fighter aircraft are designed to be unstable to give high manoeuvrability and must be flown by computer. Sophisticated software and computational fluid dynamical (CFD) packages are available for this and it may be possible to modify the code and techniques to enable them to be applied to submarines.

To improve on the linear method, further modelling of the forces and moments due to fluid effects will be necessary. Some of this may have to be done using a CFD approach. The general equations for the fluid plus submarine will then have to be treated numerically, possibly after simplification using perturbation methods.

A short discussion on the second problem of instability during surfacing led to various suggestions. It is the presence of the extra water in the conning tower which is detrimental as it causes the centre of gravity to be above the centre of buoyancy, leading to a roll moment. Thus, the possibility of installing pumps or adding extra holes at the base of the conning tower to aid the draining of water were mentioned. The addition of baffles parallel to the x-axis to change the flow and sloshing modes was also proposed, but it was unclear whether this would make the situation better or worse. Some further work was done after the conference on drainage rates from the conning tower and the magnitude of the turning moment produced by the water level in the sail being above sea–level. (See separate report). Additionally, concern was raised about whether the free-flood casing meant that the submarine behaviour was significantly different from a rigid body during normal manoeuvring as well as on surfacing. More work is needed in this area to establish whether the effects are important or not.

References

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Appendix

Let $\mathbf{R}$ be the position vector of a point P in space with respect to inertial axes fixed in space, and $\mathbf{R_0}$ be the position of the centre of the sub–axes. Then $\mathbf{r} = \mathbf{R} - \mathbf{R_0}$ is the position vector of P in the sub–axes. The submarine is rotating with angular velocity $\boldsymbol{\Omega}$, which is given by

$$\boldsymbol{\Omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \tag{A.1}$$

where p, q and r are the angular velocities about the x, y and z axes respectively. Then given any vector quantity $\mathbf{q}$, the rate of change of $\mathbf{q}$ with respect to time in the inertial fixed frame is related to that in the moving sub-frame of reference by

$$\left(\frac{d\mathbf{q}}{dt}\right)_{in} = \dot{\mathbf{q}} + \boldsymbol{\Omega} \times \mathbf{q}, \tag{A.2}$$

where '$\cdot$' is the time derivative in the moving frame (Fowles 1977). Thus applying this to the vector $\mathbf{R} - \mathbf{R_0}$, we have

$$\left(\frac{d\mathbf{R}}{dt}\right)_{in} - \mathbf{U}_0 = \dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}, \tag{A.3}$$

where $\mathbf{U}_0$ is the velocity of the origin of the sub-frame. Differentiating with respect to time again and noting that

$$\left(\frac{d\mathbf{U}_0}{dt}\right)_{in} = \dot{\mathbf{U}}_0 + \boldsymbol{\Omega} \times \mathbf{U}_0, \tag{A.4}$$

we obtain

$$\left(\frac{d^2\mathbf{R}}{dt^2}\right)_{in} = \ddot{\mathbf{r}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \dot{\mathbf{U}}_0 + \boldsymbol{\Omega} \times \mathbf{U}_0. \tag{A.5}$$

If P is a point of the rigid body, then the time derivatives of $\mathbf{r}$ in the sub-frame are zero. By taking $\mathbf{R}_g$ to be the position of the centre of gravity in the inertial frame and $\mathbf{r}_g$ to be the corresponding position in the sub-frame, we can apply Newton's second law

$$\mathbf{F} = m\left(\frac{d^2\mathbf{R_g}}{dt^2}\right)_{in}, \tag{A.6}$$

to obtain

$$\mathbf{F} = m\left[\dot{\boldsymbol{\Omega}} \times \mathbf{r}_g + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_g) + \dot{\mathbf{U}}_0 + \boldsymbol{\Omega} \times \mathbf{U}_0\right]. \tag{A.7}$$

Resolving $\mathbf{F}$, $\mathbf{U}_0$, and $\mathbf{r}_g$ in sub-coordinates as

$$\mathbf{F} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \mathbf{r}_g = \begin{pmatrix} x_g \\ y_g \\ z_g \end{pmatrix}, \tag{A.8}$$

we obtain the equations

$$
\begin{aligned}
X &= m\left[\dot u - vr + qw - x_g(q^2+r^2) + y_g(qp-\dot r) + z_g(pr+\dot q)\right],\\
Y &= m\left[\dot v + ru - pw + x_g(pq+\dot r) - y_g(p^2+r^2) + z_g(rq-\dot p)\right],\\
Z &= m\left[\dot w + pv - qu + x_g(pr-\dot q) + y_g(qr+\dot p) - z_g(p^2+q^2)\right].
\end{aligned}
\tag{A.9}
$$

These agree with the general equations of motion provided.

We now look at the moments acting on the body. The angular momentum, $\mathbf{L}$ about the centre of gravity can be written in terms of moments and products of inertia as

$$
\mathbf{L} = \begin{pmatrix} I'_x & -I'_{xy} & -I'_{xz} \\ -I'_{yx} & I'_y & -I'_{yz} \\ -I'_{zx} & -I'_{zy} & I'_z \end{pmatrix} \boldsymbol{\Omega}. \tag{A.10}
$$

We note that here a typical moment and product of inertia are defined as

$$
I'_x = \int \rho(y'^2 + z'^2)\,dV, \qquad I'_{xy} = \int \rho x' y'\,dV, \tag{A.11}
$$

where x', y', z' are the distances from the centre of gravity, so that the product of inertia is the negative of the usual definition. If $\mathbf{N}'$ is the moment of the external forces about the centre of gravity, then

$$
\left(\frac{d\mathbf{L}}{dt}\right)_{in} = \mathbf{N}'. \tag{A.12}
$$

But

$$
\left(\frac{d\mathbf{L}}{dt}\right)_{in} = \dot{\mathbf{L}} + \boldsymbol{\Omega}\times\mathbf{L}, \tag{A.13}
$$

and

$$
\mathbf{N}' = \mathbf{N} - \mathbf{r}_g \times \mathbf{F}, \tag{A.14}
$$

where $\mathbf{N}$ is the moment about the origin in the sub-frame. Thus using $\mathbf{F}$ from $(A.7)$ and defining the components of $\mathbf{N}$ to be K, M and N, we obtain

$$
\begin{aligned}
K &= I_x\dot p + (I_z - I_y)qr - I_{zx}(\dot r + pq) + I_{yz}(r^2 - q^2) + I_{xy}(pr - \dot q)\\
&\quad + my_g(\dot w + vp - qu) - mz_g(\dot v - wp + ru),\\
M &= I_y\dot q + (I_x - I_z)rp - I_{xy}(\dot p + qr) + I_{zx}(p^2 - r^2) + I_{yz}(qp - \dot r)\\
&\quad + mz_g(\dot u - vr + wq) - mx_g(\dot w - uq + vp),\\
N &= I_z\dot r + (I_y - I_x)pq - I_{yz}(\dot q + rp) + I_{xy}(q^2 - p^2) + I_{zx}(rq - \dot p)\\
&\quad + mx_g(\dot v - wp + ur) - my_g(\dot u - vr + wq),
\end{aligned}
\tag{A.15}
$$

where

$$
\begin{aligned}
I_x &= I'_x + m(y_g^2 + z_g^2),\\
I_{xy} &= I'_{xy} + mx_g y_g,
\end{aligned}
$$

etc. so that I_x, I_{xy}, ... are the moments and products of inertia about axes through the origin in the sub-frame.

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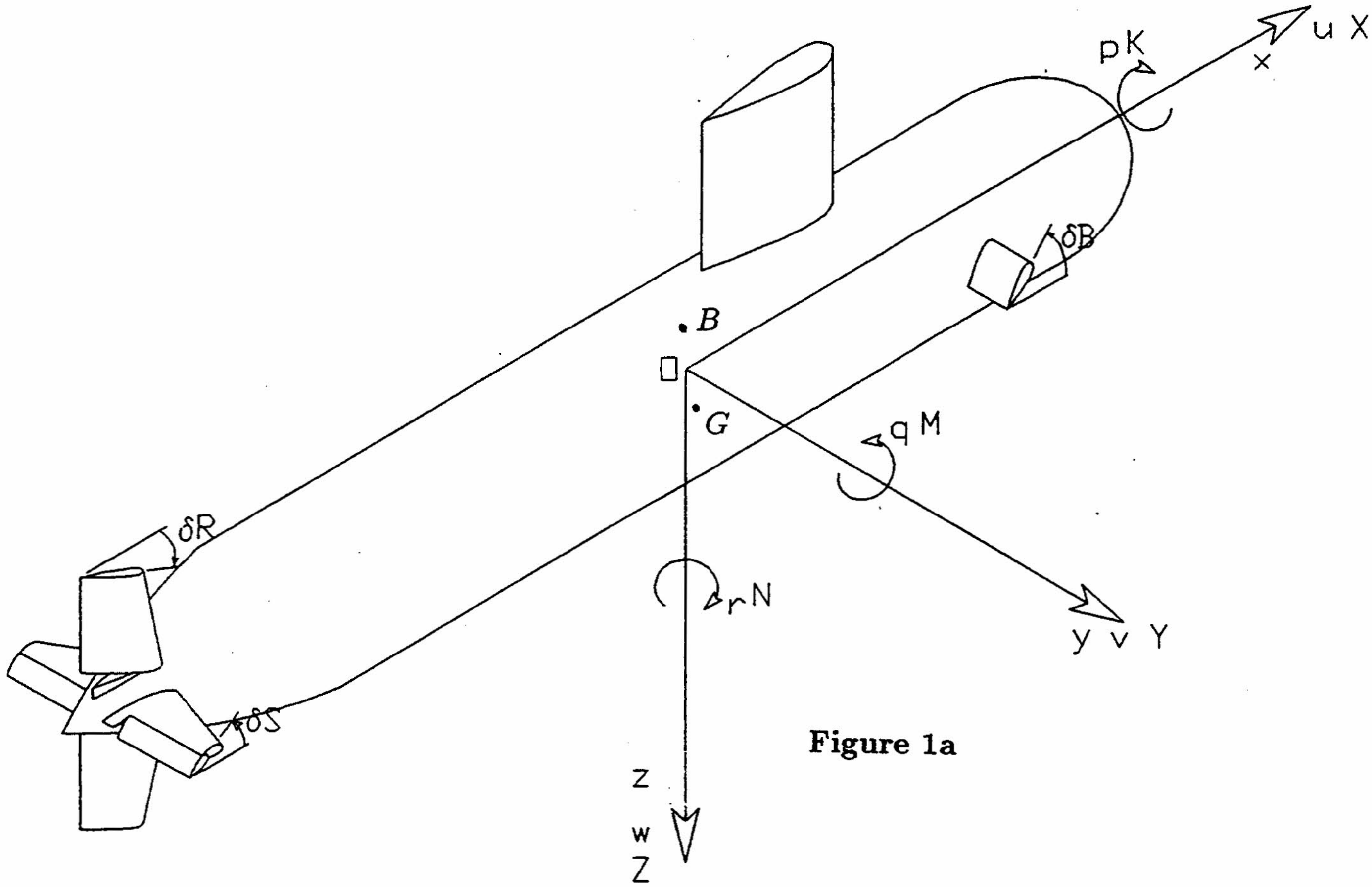


Figure 1a

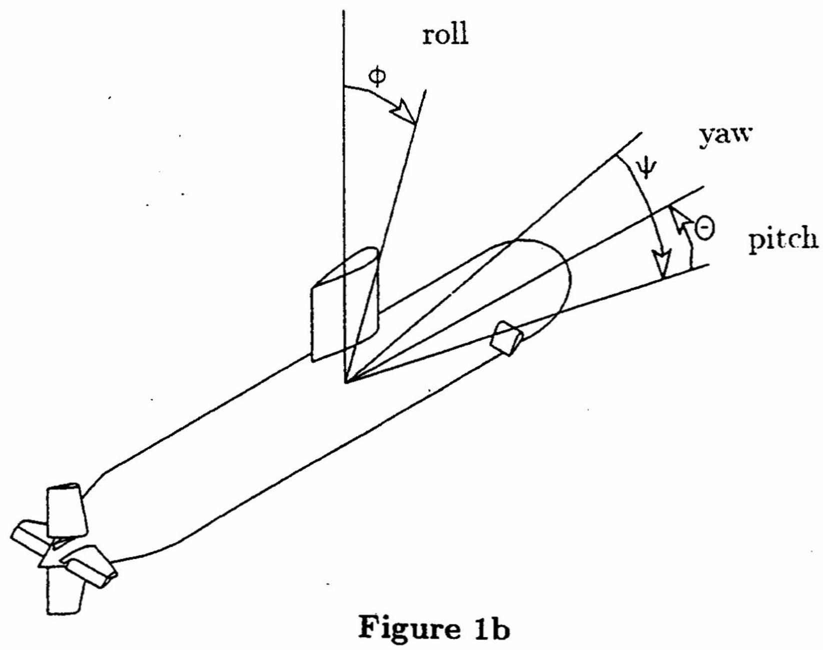


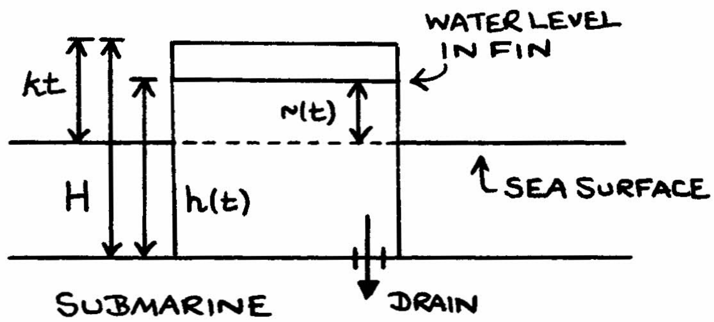
Figure 1b

NOTE ABOUT THE SUBMARINE SURFACING PROBLEM

I was not present at the 1994 Study Group meeting at Strathclyde, so I didn't hear the background to this problem, but, as a former RNVR National Service Officer who served in submarines in the years 1953 - 1955, I wonder if this is a new manifestation of an old problem. In the famous book "One of Our Submarines" by Edward Young D.S.O., D.S.C., R.N.V.R., (Rupert Hart-Davis, 1952) p.68 one reads:-

"Surfacing in really rough weather was a frightening business, there was a tricky moment before we reached full buoyancy, and before the water had drained out of the bridge casing, when the boat was extremely unstable, and if we surfaced with our beam to the waves there was a serious danger of being rolled right over. These early S-boats were particular tender in this respect, and one of them is believed to have been lost through turning turtle."

In what follows I calculate how much sea-water might still be left in the "fin" of a modern submarine when the base of the fin clears the surface. This will depend on the drainage arrangements for the fin, the key parameters being A_1/A_2, the ratio of A_1 the (horizontal plane) cross-sectional area of the fin and A_2 the total area of the drainage outlets, k the vertical rate of surfacing of the submarine as the fin rises above the surface of the sea, and H the height of the fin. The danger that this top-weight of water represents can be calculated using estimates of the submarine hull diameter and of the submarine's total weight and metacentric height. Estimates here have been made from published information and photographs in popular books e.g. of the "Swiftsure" class of fleet submarines - "SSNs". The basic physical law used is Torricelli's formula in which the flow rate (m^3/s) through an orifice of area A is given by $\alpha A\sqrt{2gh}$ where α is a "coefficient of contraction" (taken in this note as unity), g the acceleration due to gravity ($9.81 m/s^2$) and h denotes the head of water above the orifice. (This formula with $\alpha = 1$ is used extensively in the "Admiralty Manual of Seamanship" to calculate flooding of a ship which has suffered damage from holing due to shell fire).



Consider the geometry as shown in Fig. 1. Considering the drainage from the fin using Torricelli's rule

$$-A_1\dot{h} = \alpha A_2\sqrt{2gr}, \tag{1}$$

$r(t)$ being the hydrostatic head (in a "quasi-steady" calculation). Now since $h(t)-r(t)+kt = H$ then $\dot{r} = \dot{h}+k$, so that

$$-A_1(\dot{r}-k) = \alpha A_2\sqrt{2gr} \tag{2}$$

or

$$\dot{r} = k - \frac{\alpha A_2}{A_1}\sqrt{2gr}\ . \tag{3}$$

Now put $y^2 = r$ so that $\dot{r} = 2y\dot{y}$ then substituting into (3) we obtain

$$\dot{y} = \frac{k}{2y} - \frac{\alpha A_2}{A_1}\sqrt{\frac{g}{2}} \tag{4}$$

or

$$\int \frac{y\,dy}{(1-\frac{2\alpha A_2}{kA_1}\sqrt{\frac{g}{2}}y)} = \int \frac{k}{2}dt, \tag{5}$$

which may be integrated to give

$$\frac{1}{2}\left(\frac{A_1}{A_2}\right)^2\frac{k^2}{\alpha^2 g}\left\{\left[1-\frac{2\alpha}{k}\left(\frac{A_2}{A_1}\right)\sqrt{\frac{g}{2}y}\right]-\ln\left[1-\frac{2\alpha}{k}\left(\frac{A_2}{A_1}\right)\sqrt{\frac{g}{2}y}\right]\right\} = \frac{k}{2}t + C\ . \tag{6}$$

Now when $t = 0, h(0) = H$ so $r(0) = 0$, hence $y(0) = 0$ and $C = \frac{1}{2}\left(\frac{A_1}{A_2}\right)^2 \frac{k^2}{\alpha^2 g}$. Then equation (6) can be rewritten as

$$\left[1 - \frac{2\alpha}{k}\left(\frac{A_2}{A_1}\right)\sqrt{\frac{g}{2}}y\right] - \ln\left[1 - \frac{2\alpha}{k}\left(\frac{A_2}{A_1}\right)\sqrt{\frac{g}{2}}y\right] = \alpha^2\left(\frac{A_2}{A_1}\right)^2 \frac{g}{k}t + 1 \tag{7}$$

or

$$Y(t) - \ln Y(t) = Kt + 1 \ , \tag{8}$$

where

$$Y(t) = 1 - \frac{2\alpha}{k}\left(\frac{A_2}{A_1}\right)\sqrt{\frac{g}{2}}y \ , \quad K = \alpha^2\left(\frac{A_2}{A_1}\right)^2 \frac{g}{k} \ . \tag{9}$$

Calculation of water height in the fin on completion of surfacing.

We consider surfacing to be complete when the fin base clears the surface of the sea i.e. at a time $t = t^* = H/k$.

We seek to find the height of water in the fin which is $r(t^*)$. We first find $y(t^*)$ from solving equation (8) for $Y(t^*)$. This can be done iteratively using a pocket calculator knowing that the right-hand side of (8) is $1 + Kt^* = 1 + \alpha^2\left(\frac{A_2}{A_1}\right)^2\frac{gH}{k^2}$. From equation (9) we can calculate $y(t^*)$ from $Y(t^*)$ and finally $r(t^*)$ since $r(t^*) = y^2(t^*)$ by the definition of r.

In the subsequent table of results H has been taken from drawings of the "Swiftsure" class of SSN fleet submarines as $H = 10m$, $\alpha = 1$, $g = 9.81 m/s^2$ and a range of values of k (m/s), the vertical rate of surfacing, and of A_1/A_2, the ratio of fin horizontal cross-sectional area (A_1) to the total area of drainage orifices (A_2) have been considered. The results are tabulated below in Table 1.

k	0	1	2	3	∞
A_1/A_2					
20	0	6.10	7.87	8.49	10.00
10	0	3.58	6.11	7.24	10.00

The entries in Table 1 show the height of water (m) still in the fin on completion of surfacing. (Note that improved drainage means bigger values of A_2 and smaller values of A_1/A_2.)

With further estimates of heights of this top-weight of unwanted water above the submarine centre of gravity the "overturning moment" can be compared with the "restoring moment" dependent on the metacentric height. For the "Swiftsure" class some rough estimates for the 6.10 m entry in Table 1 give an overturning moment of 989,535 kg m against a restoring moment of 1,400,000 kg m. (Assumptions: height of fin $H = 10m$, diameter of pressure hull 15 m, metacentric height 1/3 m, fin cross-sectional area, less area taken up by tower, periscopes etc. $A_1 = 15m^2$, published displacement 4,200 tonnes.)

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