Mobile Phones

BT

1 Introduction

The basic problem is to solve the two-dimensional scalar Helmholtz equation for a point source (the antenna) situated in the vicinity of an array of scatterers (such as the houses and any other relevant objects in 1 square km of urban environment). The wavelength is a few cm and the houses a few metres across, so there are three disparate length scales in the problem.

The question posed by BT concerned ray counting on the assumptions that
(i) rays were subject to a reflection coefficient of about 0.5 when bouncing off a house wall and
(ii) that diffraction at corners reduced their energy by 90%. The quantity of particular interest was the number of rays that need to be accounted for at any particular point in order for those neglected to only contribute 10% of the field at that point; a secondary question concerned the use of rays to predict regions where the field was less than 1% of that in the region directly illuminated by the antenna.

The progress made in answering these two questions is described in the next two sections and possibly useful representations of the solution of the Helmholtz equations in terms other than rays are given in the final section.

2 Ray Counting

There will in general be many rays from a transmitter T through a set of buildings to an observer O. We wish to calculate the amplitude resulting from these rays to a prescribed accuracy by truncating the sum so that it only includes those rays that have undergone at most N diffractions or reflections. If we ignore diffracted rays and assume that all reflection coefficients ρ satisfy $|\rho| \leq \rho_{\max}$ then the energy in a ray reflected n times is at most $\rho_{\max}^{2n}$ times the transmitted energy. (We ignore the effects of spherical spreading, as advised in [1].) Since the requirement is to get an upper bound on the total contribution of the omitted rays *without* actually calculating them, this factor of $\rho_{\max}^{2n}$ is the most we can say about the intensities. Now let R_n = number of rays propagating from T to O via exactly n reflections. Then the total power in the omitted rays, as a fraction of the transmitted power, is at most

$$\sum_{n=N+1}^{\infty} R_n \rho_{\max}^{2n} \tag{1}$$

and we wish to ensure that this is at most 10% (or some other specified tolerance.) In order to do this, we have to obtain an upper bound on R_n and the problem is that most of the natural bounds that one can write down are exponential in n, and will therefore not guarantee convergence of (1). For instance, suppose that from each wall of a building at most N_w walls of other buildings are visible, with a value of N_w that is a fixed upper bound, valid for all buildings in the area of interest. Then it is straightforward that R_n grows no faster than N_w^n, because after each reflection there are at most N_w walls to choose as the next wall to reflect from. But this is not enough to cause (1) to converge unless $N_w \rho_{\max}^2$ happens to be less than 1.

We therefore consider a relatively simple but not untypical geometry of buildings, and we wish to obtain for it a bound on R_n that ensures convergence of (1) for all $\rho_{\max} < 1$: a polynomial in n would be satisfactory.

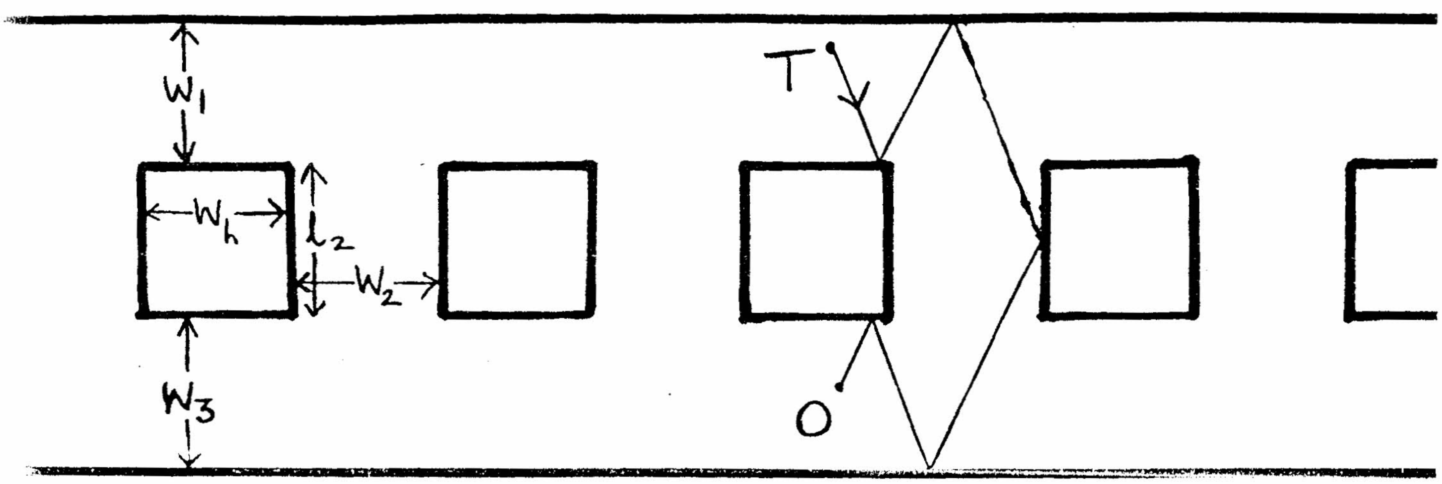

Figure 1: Paradigm configuration of buildings

We consider a regular infinite row of equal rectangular buildings between parallel infinite walls, as in figure (1). This could represent a row of office blocks separated by cross streets between two major roads. Alternatively, it could represent a row of houses separated by side passages leading from the street area to the back garden area. We take the dimensions and terminology to be as in figure (1) with

w_1 = width of the north corridor

w_2 = width of the gaps

l_2 = length of the gaps = y–dimension of buildings

w_3 = width of the south corridor

w_h = width (x–dimension) of the buildings.

The typical situation here is that the transmitter T will be in say the north corridor, and it is necessary to consider observer positions O throughout the diagram. For O in the north corridor there is not a difficulty, and for O in one of the gaps the results of the following section give the necessary information. We therefore consider the case where O is in the south corridor. In such cases, we understand that it would generally not be necessary to count rays that make more than one passage down one of the gaps. Omitting those, we are left to count rays that make say n_1 reflections in the north corridor, then n_2 reflections in one of the gaps, then n_3 in the south corridor and finally arrive at O. For instance, figure (1) shows a ray with $(n_1, n_2, n_3) = (2, 1, 2)$.

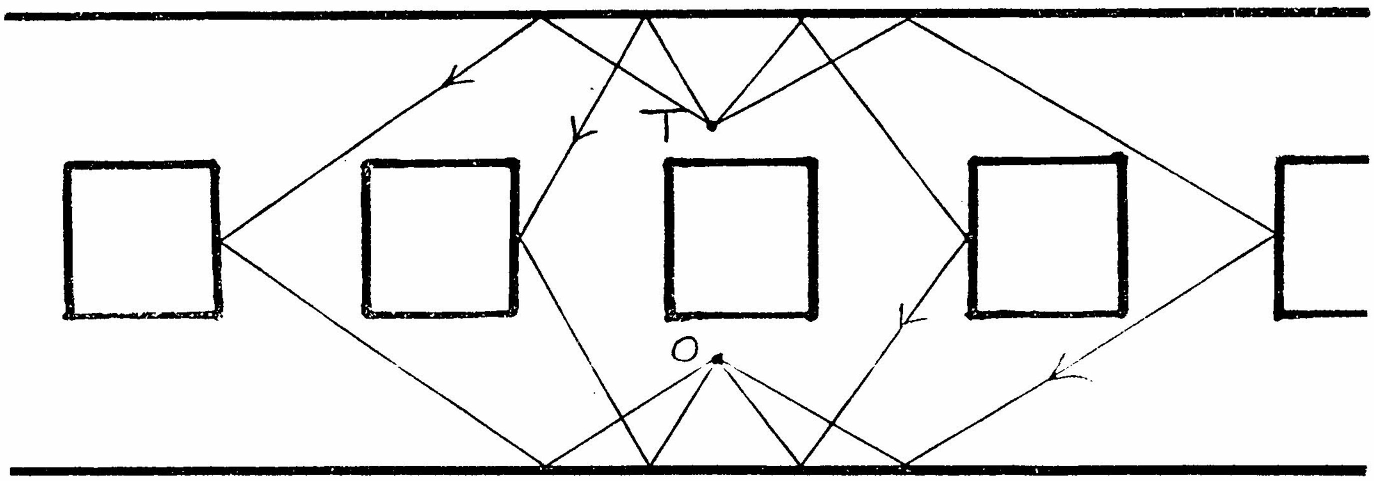

Figure 2: Four ray, four gap example.

There are exactly $(n + 1)(n + 2)/2$ triples (n_1, n_2, n_3) of non-negative integers satisfying $n = n_1 + n_2 + n_3$ but there will be more than one ray from T to O with a given triple. For instance figure (2) shows 4 rays from T to O, all with $(n_1, n_2, n_3) = (1, 1, 1)$, using four different gaps. A satisfactory bound on R_n can be obtained as follows :

We first obtain an upper bound for rays where the first and last of the n_2 reflections are from the east wall of the gap used. For these we show that
(a) Given a particular choice of gap, there is a most one ray with the given (n_1, n_2, n_3) parameters.
(b) Given (n_1, n_2, n_3) there is a bound on the number of gaps that the ray can use.

To see these, think of a ray that leaves T at an acute angle $\pm\theta$ to the eastgoing corridor axis. It travels for an eastwards distance l_1 involving n_1 reflections. It then enters one of the gaps, travelling at angles $\pm(\pi/2-\theta)$ to the southward axis, making n_2 reflections on its passage south through the gap. It then travels a distance l_3 west to reach O after a further n_3 reflections. To accomplish this, the n_1 reflections alternately off the north and south walls of the north corridor must end with a reflection off the north wall. The n_2 reflections start and finish on the east wall of the gap so n_2 is odd. The n_3 reflections must start on the south wall of the south corridor. Hence the reflections occur in a known order from known reflectors; so the position of the n–fold reflected image of T is known; so the angle of launch to reach O is known, establishing (a). For (b) we put down some inequalities on the lengths. The x–interval between successive reflections in the north corridor is $w_1/\tan\theta$. So if the ray makes precisely n_1 reflections while travelling along that corridor, leaving it a distance l_1 east of T, then

$$n_1 - 1 \leq \frac{l_1}{(w_1/\tan\theta)} \leq n_1 + 1. \tag{2}$$

Similarly for traversing the gap we have

$$n_2 - 1 \leq \frac{l_2}{w_2 \tan\theta} \leq n_2 + 1, \tag{3}$$

and for the south corridor

$$n_3 - 1 \leq \frac{l_3}{(w_3/\tan\theta)} \leq n_3 + 1. \tag{4}$$

Multiplying (2)by (3) we have

$$(n_1 - 1)(n_2 - 1) \leq \frac{l_1 l_2}{w_1 w_2} \leq (n_1 + 1)(n_2 + 1). \tag{5}$$

Thus the value of l_1 is restricted to a range of

$$l_{1+} - l_{1-} = 2(n_1 + n_2) w_1 w_2 / l_2. \tag{6}$$

However, a restriction on the range of l_1 restricts the number of gaps that a ray with this (n_1, n_2, n_3) can use. In fact, a range $[l_{1-}, l_{1+}]$ allows at most

$$\frac{(l_{1+} - l_{1-}) + w_2}{w_h + w_2} + 1 \tag{7}$$

gaps to be used. Substituting (6) into this we see that the number of usable gaps is at most

$$c_1(n_1+n_2)+d \tag{8}$$

where

$$c_1=\frac{2w_1w_2}{l_2(w_h+w_2)}, \qquad d=\frac{w_2}{w_h+w_2}+1. \tag{9}$$

The sum of (8) over *odd* values of n_2 for a fixed value of $r=n_1+n_2$ is

$$(c_1r+d)\left\lfloor\frac{r+1}{2}\right\rfloor. \tag{10}$$

Removing the $\lfloor\ \rfloor$ makes a slight increase, and then summing over r from 0 to n gives at most

$$c_1\frac{n(n+1)(n+2)}{6}+d\frac{(n+1)(n+2)}{2}. \tag{11}$$

In fact, a similar bound could be obtained by arguing with the south corridor in place of the north, so the factor of c_1 from (9) here can be replaced by

$$c=\frac{2\min(w_1,w_3)w_2}{l_2(w_h+w_2)} \tag{12}$$

to give a better bound if $w_3<w_1$.

Then doubling this to include rays whose first and last n_2 reflections are from the *west* wall of the gap used, we have at most

$$cn(n+1)(n+2)/3+d(n+1)(n+2) \tag{13}$$

rays with odd n_2.

It remains to add in a bound on the number of rays with n_2 even. These are covered by (a) or (b):

(a) T and O are opposite the same gap : in such a case, n_1 and n_3 must each be 0 or 1, with n_1+n_3 of the same parity as n. For each of the allowed possibilities for (n_1,n_3), there may be two different rays, according as the first of the n_2 reflections is on the east or west wall of the gap. In any case there are at most 4 relevant rays if $n\geq 2$, 2 if $n=1$, and 1 if $n=0$.

(b) When T and O are not opposite the same gap, suppose say that O is east of T. The rays must make the first of their n_2 reflections from a west-facing wall, and so the order of the reflections is completely determined by (n_1,n_2,n_3), except for the fact that we have not specified which gap the ray goes through. However, the combined effect of the n_2 reflections on the image of T is the same, whichever gap it is, so in fact the value of θ is uniquely determined by (n_1,n_2,n_3) in such a case, and therefore the gap is uniquely determined. So the number of such rays is at most the number of (n_1,n_2,n_3) with n_2 even and $n_1+n_2+n_3=n$, which is at most

$(n/2+1)^2$. This always exceeds the upper bound in case (a), so it suffices to use this latter form as the upper bound. We therefore have

$$R_n \leq cn(n+1)(n+2)/3 + d(n+1)(n+2) + (n/2+1)^2. \tag{14}$$

This is undoubtedly quite a weak upper bound — R_n will not really be as large as $O(n^3)$ — but it is enough to ensure convergence of (1) which was our aim. For any particular dimensions of the buildings and corridor widths, and a given ρ_{max}, the value of N required can be computed explicitly from this. For instance, taking the case

$w_2 = w_h$ (building and gap widths equal);

$\min(w_1, w_3) = l_2$ (one corridor width equal to gap length, the other wider);

$\rho_{\mathrm{max}} = 1/2$;

it turns out that $N = 6$ is enough to make (1) less than 10%. We might even conjecture that in general (1) will be $O(N^3/2^N)$.

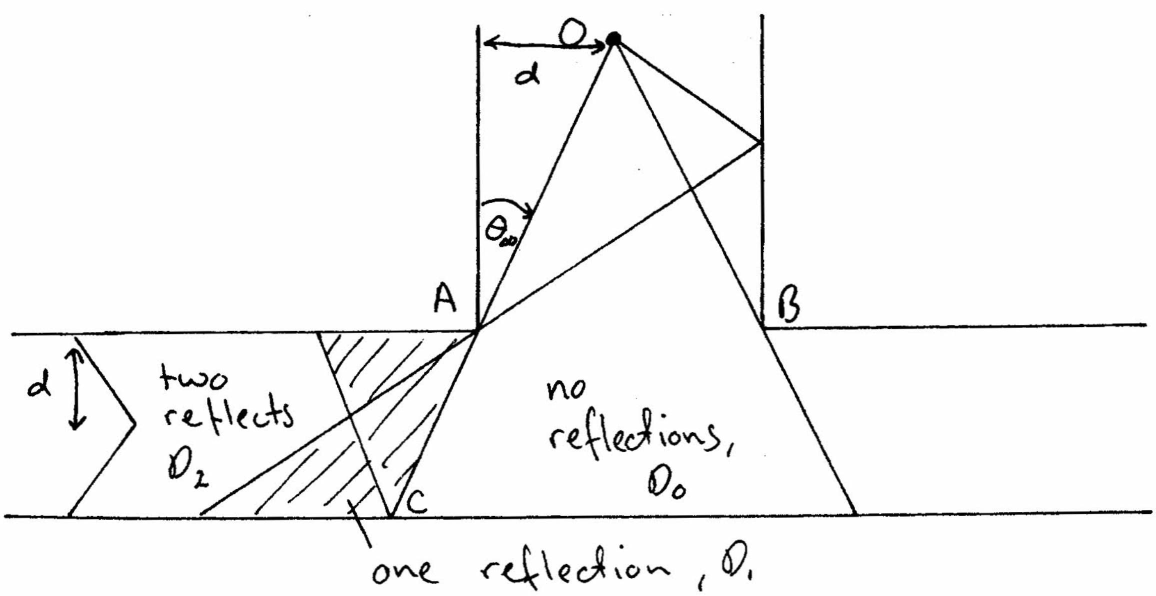

Figure 3: T-junction paradigm

3 Intensity Estimation

Even for relatively simple configurations of houses, counting all the rays through every point is time consuming. However, we can get a rough 'intensity map' by ignoring diffraction and dividing the domain up into regions which can be reached by 1,2,3,... reflections; the region which can only be reached by 3 reflections will have roughly half the intensity of that which can be reached by only 2 reflections, if the reflection coefficient is 1/2.

A paradigm problem for this intensity map is the T-junction:

To calculate the boundaries of our intensity regions we need to calculate the rays which pass through the corners A,B. The boundary of the region which can be reached without reflection is simply the shadow boundary OAC. To calculate the boundary of the region which can be reached by 1 reflection we need to calculate the rays with one reflection passing through A. The distance the ray travels down the side road after a further n-m reflections is then $x = (2(n-m)+2)d\tan\theta_{nm}$ i.e.

$$x_{nm} = (2(n-m)+2)\frac{d^2}{h}(2n+1) \tag{15}$$

By maximising this distance with respect to m we can find out which rays propagate furthest down the side road and we find x_{nm} is maximum when $m = n/2 - 1/4$. Thus the rays which propagate furthest down will correspond to integer values of m on either side of this maximum. We see that these will have roughly the same number of bounces before and after the corner. Our final intensity map will look something like figure 4

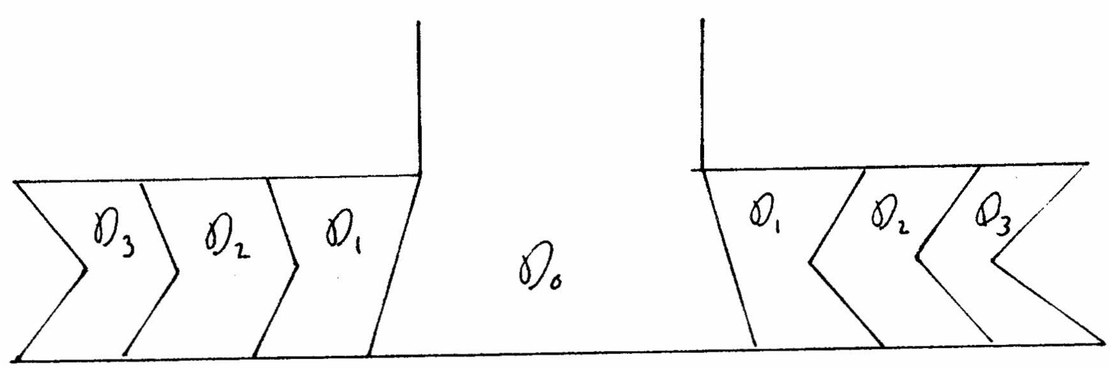

Figure 4: Final intensity map

The method can be extended, in principle, to more complicated domains. With a second junction, as shown in figure 5. The rays forming the boundaries of the regions D_n may either pass through A or E, and thus we will need to consider the rays through each of these points.

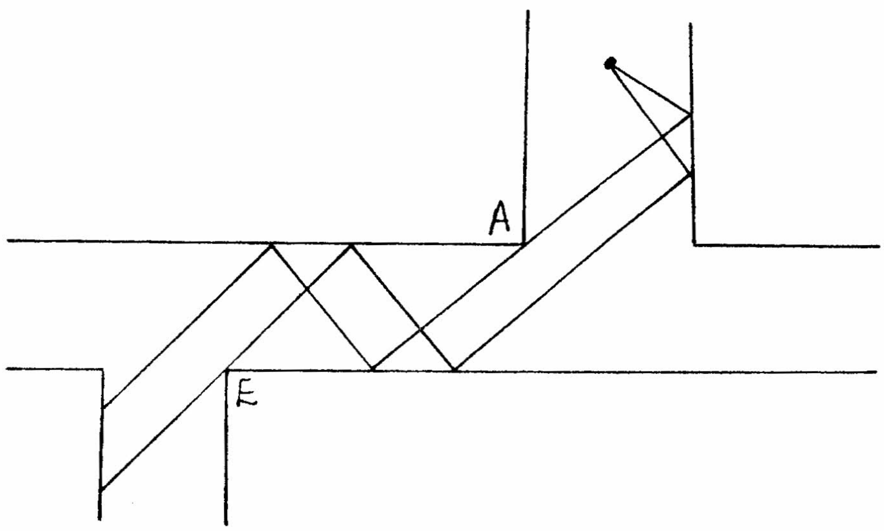

Figure 5: Two junction problem

4 Alternative representations of solutions of Helmholtz Equation

There are several other approaches besides that of geometrical optics that may be considered, depending on the structure of the environment. In particular, for housing distributions that are either periodic or nearly random then there may be the possibility of constructing 'homogenised' models that give a quick estimate of the average field strengths.
(i) Nearly Periodic Environment: WKB expansions and generalisations

Several suggestions were made concerning the use of WKB approximations when many large-wavelength-scale scatterers are present. These can be illustrated with reference to the
a) Paradigm o.d.e.

$$\frac{d^2\phi}{dx^2} + k^2 f(x)\phi = 0 \ , \quad -L < x < L \tag{16}$$

where $f > O$ is a function with period 1 (house scale) with slow modulation over a large 'estate' length scale $L(1km)$; $k \gg 1$ is essentially the inverse of the wavelength on the house scale. When $L = \infty$ it is usual to write

$$\phi \sim A \exp(\pm iku) \tag{17}$$

where $u'^2 = f(x)$ and $A \propto 1/f^{1/4}$, the signs being chosen to satisfy appropriate radiation conditions. This WKB representation is the basis for ray theory but it has the disadvantage of neglecting all reflections from the potential $f(x)$ (the reflection coefficient of an isolated scatterer is exponentially small as $k \to \infty$). Although it can thus only describe situations where nearly all the energy is being transmitted through the medium, it can be modified to account for modulations on an L scale caused by slowly varying f by introducing a 'multiple scale' ansatz

$$\phi \sim A(x, X) \exp(iku(x, X))$$

where, say, $x = LX$. A particularly interesting regime from the mathematical viewpoint occurs when $L \geq O(k)$, when the modulation can cause ku to change by $O(1)$. The introduction of boundary conditions at $x = \pm L$ (the edge of the estate) causes waves to travel in both directions and the representation (17) can be used estimate the high eigenfrequencies in such an estate.

The introduction of an 'antenna' at $x = 0$, say could be discussed in terms of the paradigm

$$\frac{d^2\phi}{dx^2} + k^2 f(x)\phi = \delta(x) , \quad -L < x < L \tag{18}$$

but the presence of the localised forcing invalidates the use of the WKB approximation near the origin. It may be possible to get round this by constructing some kind of localised expansion near $x = 0$; also it was pointed out that the solution of (18) can always be represented via a Finite Fourier transform as

$$\phi(x) = \Sigma a_n \phi_n(x) \tag{19}$$

where

$$\phi_n'' + k^2 f \phi_n = \lambda_n \phi_n \tag{20}$$

with $|\phi_n| = 1$ and suitable boundary conditions at $x = \pm L$ and

$$a_n = \int_{-L}^{L} \phi_n \phi dx = \phi_n(0)/\lambda_n.$$

It was also pointed out that any WKB representation relies on f having no abrupt changes on a scale of $x \sim O(k^{-1})$, for exactly the same reasons that the δ-function in (18) would invalidate (17).

Finally it was realised that all the above discussion is closely related to the idea of Anderson localisation and the questions of how much change in A and u occur over one period (or 'quasi'-period) of f [2], [3], [4]. Hence, delicate questions concerning the limiting behaviour of discretisations and 'propagator matrices' may well arise, whose answers will probably differ from similar questions posed in 2-D problems.

b) 2-D problems

While many of the above ideas can be easily generalised to the 2-D version

$$(\nabla^2 + k^2)\phi = \delta(x)\delta(y) \qquad -L < x < L,\ -L < y < L \tag{21}$$

with reflection/diffraction conditions on a house in each cell $n < x < n+1$, $m < y < m+1$ (maybe with a slow modulation), the extra spatial dimension may allow or demand representations within a cell different from the simple formula (17). In particular it was proposed that use might be made of the 'hybrid method' [5] proposed for wave propagation in slowly varying waveguides. Crudely speaking the solution of (21) with say $\pm\partial\phi/\partial y + \alpha(x)\phi = 0$ (α nearly periodic on $O(1)$ scale), on $y = \pm 1$ is represented as

$$\phi = \exp(i\lambda x)\psi(x, y) \tag{22}$$

where in the unforced problem, ψ and λ satisfy an eigenvalue problem in y. Hence in (21) ψ can be represented as an eigenfunction expansion and the idea behind the hybrid method is to retain only the lower terms in this expansion, the higher terms being converted into a contour integral by using Watson's transforms and hence, via a steepest descents analysis, into a ray representation. However the details of the construction of the eigenfunction expansion so as to incorporate the antenna remain very unclear.

In the 'periodic housing estate' problem (21), the hybrid method may involve generalising (22) to

$$\phi = \exp(i\lambda\underline{\zeta}\cdot\underline{x})\psi \qquad |\underline{\zeta}| = 1 \tag{23}$$

and integrating over the direction $\underline{\zeta}$ (i.e. subjecting each house to plane wave irradiation), but again the implementation details are unclear.

(ii) Random Environments

When the environment comprises many random scatterers it was suggested that one crude modelling approach would be to follow that of radiative heat transfer in the 'optically thick' limit [6]. The argument might go as follows: Let $q(\underline{x}, \underline{\zeta})$ be the e.m. energy flux at $\underline{x}$ in direction $\underline{\zeta}$, so that $\int\int \underline{q}\cdot\underline{\zeta} ds = E(x)$ is the local energy density. Also let $l(x) = O(1)$ be the local length-scale over which radiation is scattered and absorbed (house length) and L be the estate scale as before. In the optically thick limit $L \gg 1$. Also assume the absorption coefficient, *i.e.* the proportion of the energy, E, absorbed per scatterer is $\rho(x) = O(\delta)$ say. Then the variation of $\underline{q}$ in the direction $\underline{\zeta}$ is given by

$$\underline{\zeta}.\nabla q = -\frac{q}{l} + \frac{(1-\delta)}{4\pi l}E\,,$$

where ∇ is defined on the l-scale. Hence, on the L-scale,

$$q = \frac{(1-\delta)}{4\pi}E - \frac{l}{L}(\underline{\zeta}\cdot\overline{\nabla})q$$

We now assume that $\delta \ll 1$ and solve for q iteratively to give

$$q = \frac{(1-\delta)}{4\pi}E - \frac{l}{4\pi L}(\underline{\zeta}\cdot\overline{\nabla})E + \left(\frac{l}{L^2}\right)\frac{1}{4\pi}\underline{\zeta}\cdot\overline{\nabla}(l\underline{\zeta}\cdot\nabla)E$$

and integrate over the unit sphere in $\underline{\zeta}$ space to give

$$E \sim (1-\delta)E + \frac{l}{3L^2}\overline{\nabla}(l\overline{\nabla}\cdot E)$$

(note $\int \underline{\zeta} \cdot ds = 4\pi$, $\int \underline{\zeta} \cdot (\underline{\zeta}.ds) = \underline{0}$)

Thus if we finally make the optically thick assumption that $\delta = O(l/L)^2$, we obtain

$$l\overline{\nabla}(l\overline{\nabla}E) = \overline{p}E$$

for a suitably scaled absorption coefficient $\overline{p}$: thus the energy at any point in the estate satisfies a modified Helmholtz equation!

It has recently been suggested that work by RL Weaver [7], [8] may develop this theme. Also, concerning reflection from localised random scatterers, expertise is available from Dr Mark Williams, Department of Mathematics, Sheffield, who has studied foliage microwave scattering in conjunction with RSE, Malvern. Also, inevitably Mike Berry's 1981 Ann Phys paper has been cited [9].

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