Stresses Induced by Non-Circularity in Nominally Circular Sections Under Internal Pressure

1. INTRODUCTION

C.I.G. Gases is the largest supplier of compressed gas in cylinders in Australia. As such, the company goes to considerable pains to ensure that these containers are safe for both consumers and handlers of the product. While obvious deficiencies due to mishandling occur in these cylinders, the feature of most concern at this meeting was the incidence of a small degree of noncircularity (less than 5%) in the cross-sections of these cylinders, and the effect of this on the stresses induced in the cylinders by internal pressures.

While initially, gas cylinders were thick with a relatively small payload of gas at low pressures (80 litres of oxygen per kilogram of container weight at about 7 MPa pressure) more recent development in cylinder construction has led to much higher figures (200-400 litres/kg, at up to 30 MPa). At the same time, changes in steel composition have led to a reduction in the thickness of the walls of these cylinders (typically, tube diameter of 300 mm, with wall thickness of 5-6 mm). On the other hand, cylinders of aluminium construction have remained relatively thick-walled (20-30 mm, with tube diameter as for steel).

In general, though, the increase in energy per unit volume of gas, and a general reduction in tube thickness have led to a much reduced margin for error in estimating the likelihood of cylinder failure. In the past, cylinder design codes based on perfect circularity carried such high safety margins that any errors induced by this assumption could safely be ignored. However, in the light of the current desire for greater cylinder efficiency, it is felt that this assumption may no longer be justified, and some way is sought to incorporate the effects of noncircularity into a cylinder design code.

The 'average' tangential stress is given by:

$$\bar{\sigma} = pa/(b-a) \tag{1.1}$$

where a and b are the internal and external radii and p is the internal pressure. However, the maximum tangential stress may be substantially different to $\bar{\sigma}$ due to variability in the stress through the cross section and the possible deviation from circularity. Let σ_{max} be the maximum tangential stress in a given cylinder and define λ by the relation:

$$\sigma_{max} = \bar{\sigma}(1 + \lambda) . \tag{1.2}$$

Essentially, λ is the ratio of the maximum bending stress to $\bar{\sigma}$. A number of formulae have been identified by CIG for estimating λ. They are:

$$\lambda = 6\left(\frac{\delta}{h}\right) ; \tag{1.3}$$

$$\lambda = 6\left(\frac{\delta}{h}\right) \left(1 - \frac{p}{p^*}\right)^{-1} ; \tag{1.4}$$

$$\lambda = 6\left(\frac{\delta}{h}\right) \left(1 + \frac{p}{p^*}\right)^{-1} ; \tag{1.5}$$

and

$$\lambda = 6\left(\frac{\delta}{h}\right) \left(1 + \frac{p}{4p^*}\right)^{-1} ; \tag{1.6}$$

where δ is the maximal radial deviation from out-of-round, and h is the wall thickness of the cylinder, while

$$p^* = \frac{Eh^3}{4(1-\nu^2)R^3} \tag{1.7}$$

(where E = Young's modulus, ν = Poisson's ratio, R = mean cylinder radius) is the critical pressure (the pressure at which buckling of the cylinder wall occurs) when the cylinder is subject to *external* pressure.

The sources of the formulae (1.3)-(1.6) and the assumptions behind them are of some interest. Equation (1.3) assumes a rigid-body deformation (there is no deflection of the cross-section of the cylinder under deformation [1]) while (1.4) is based upon the assumption of an external pressure acting on the cylinder (it *does* however incorporate the effects of nonrigid-body deformation [2]). The relation (1.5) is the adaptation of (1.4) for *internal* pressure [3]. Finally, (1.6) is a design criterion based on (1.3); but obviously the denominator has been modified; although the basis for this modification was not made clear in the derivation [4].

The questions brought to the study group were:

(i) Which of the estimates (1.3)-(1.6) is most appropriate.

(ii) Can these formulae be extended to thick walled cylinders.

(iii) What are the effects of non-uniform wall thickness.

After discussion, the Study Group agreed that no attempt should be made to incorporate the effects of the ends of gas cylinders into the calculations (although it was recognized that such ends could have considerable effects). Rather, the cylinders should be modelled as an infinitely long tube; thus reducing the problem of a two-dimensional one, with the added assumption of plane strain.

2. PRELIMINARY CALCULATIONS

A natural starting point for the discussion was the Lamé (or boiler maker) solution for a circular pressure vessel. The maximum hoop stress for this problem is given by (see for example [1]):

$$\sigma_{max} = \frac{p(b^2 + a^2)}{b^2 - a^2} \qquad (2.1)$$

$$= \frac{pa}{b - a}(1 + \lambda)$$

where

$$\lambda = \frac{b(b - a)}{a(b + a)} .$$

If h/a is small however we may proceed as follows. As in section 1, the average hoop stress $\bar{\sigma}$ is given by:

$$\bar{\sigma} = \frac{pa}{(b - a)} .$$

Furthermore, the azimuthal strain produced by this stress is approximately (bearing in mind that the radial stress is substantially smaller than $\bar{\sigma}$):

$$\epsilon \quad \frac{(1 - \nu^2)\bar{\sigma}}{E} .$$

The moment associated with this change in radius is

$$M = \frac{E\,h^3\,\epsilon}{6(1 - \nu^2)(a + b)}$$

$$= \frac{h^2 ap}{6(a + b)}$$

which leads to the approximation

$$\sigma_{max} = \frac{pa}{h} + \frac{p}{2} + O\left(\frac{hp}{a}\right) . \qquad (2.2)$$

It is easy to verify that (2.2) and (2.1) are in good agreement provided (h/a) is small. Note however that the deviation from the average stress is a bending stress not associated with out-of-roundness of the cylinder. Thus, bending stresses can not always be associated with a deviation from circularity.

As a second preliminary calculation, consider a cylinder of uniform thickness that a slightly out-of-round. Let the internal wall be described by:

$$r = a + \delta u(\theta)$$

when the cylinder is uncharged. For $h/a << 1$ and large internal pressure ($p >> p^*$) we expect the final shape to be essentially circular. Thus, the bending moment due to out-of-roundness will be approximately:

$$M = - \frac{p^* R^3 \kappa}{12}$$

and hence for large p,

$$\sigma_{max} \approx \frac{p(b^2 + a^2)}{h(b + a)} + 2\frac{p^* R^3 \kappa}{h^2}$$

$$\approx \frac{pa}{h}\left(1 + 2\frac{p^* R^2 \kappa}{h}\right) \qquad (2.3)$$

where

$$\kappa = \frac{\delta}{R^2} \max|\ddot{u} + u|$$

is the maximal deviation of curvature from circularity. That is, for $h/a << 1$ and $p/p^* >> 1$

$$\lambda \approx 2\,\frac{\kappa R^2 p^*}{hp}\,. \tag{2.4}$$

It is worthwhile to compare (1.3)-(1.6) with the asymptotic expression (2.4). Expression (1.3) is independent of p and is therefore inappropriate if δ is taken to be the maximal deviation in the radius of the *uncharged* cylinder. In the next section we show that (1.1) is actually quite appropriate if δ is the deviation in radius for the *charged* cylinder. If we assume that the initial shape of the cylinder is elliptical (i.e. $u = \cos 2\theta$), then equations (2.4) and (1.5) are consistent. A sign change in (1.4) to account for the fact that the pressure is internal, not external, will make the equation consistent for an elliptical cross section also. Finally, (1.6) is consistent if $\kappa = 12\delta/R^2$ but the rationale for this choice is unclear.

3. THIN WALLED VESSELS

The first step is to consider the equilibrium equations for the vessel. Consider an element of cross-section of the tube, as shown in Figure 1, of length ds. We assume that the interior boundary of this element has radius of curvature R_c and the element experiences a moment M about the midplane, shearing force S at the (radial) cross-section, hoop force H, and is subject to a (constant) internal pressure p. Figure 1 then shows this sytem of forces and moments as they act on this tube element. If one then considers the conditions for equilibrium of this element, one obtains the set of equations:

$$\frac{dM}{ds} + \left(1 + \frac{h}{2R_c}\right) S = 0 \tag{3.1}$$

$$p + \frac{dS}{ds} - \frac{H}{R_c} = 0\,, \tag{3.2}$$

$$\frac{dH}{ds} + \frac{S}{R_c} = 0\,, \tag{3.3}$$

where h is the thickness of the tube.

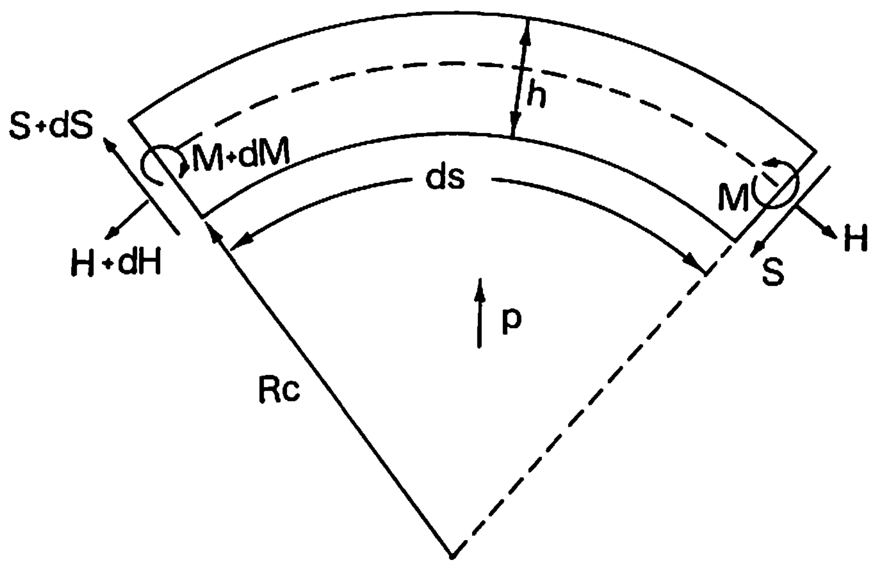

Figure 1: Element of Cylinder.

Let the inner wall of the *charged* cylinder be given by:

$$r = a + \delta v(\theta)$$

where

$$\int_0^{2\pi} v(\theta)\, d\theta = 0, \qquad \max|v| = 1.$$

Then,

$$\frac{1}{R_c} = \frac{1}{a} - \frac{\delta}{a^2}(v + v'') + O(\delta^2)\,. \tag{3.4}$$

and

$$\frac{ds}{d\theta} = a + \delta v + O(\delta^2)\,. \tag{3.5}$$

Substitution of (3.4) and (3.5) into (3.1)-(3.3) yields:

$$M = M_o + \delta R v p + O(\delta^2)$$

$$H = p\,(a + \delta v) + O(\delta^2)$$

$$S = -p\delta v' + O(\delta^2)$$

where M_o is constant. However, M_o can be estimated as in section 2. That is,

$$M_o = \frac{h^2 ap}{12R} .$$

Thus,

$$M \approx \frac{h^2 ap}{12R} + \delta R v p$$

and hence

$$\sigma_{max} \approx \frac{pa}{h}\left(1 + \frac{h}{2a} + \frac{6\delta}{h}\right) \tag{3.6}$$

$$\approx \frac{pa}{h}\left(1 + \frac{6\delta}{h}\right) . \tag{3.7}$$

Equation (3.7) corresponds precisely to (1.3) if the deviation from circularity is taken to be the deviation of the *charged* cylinder. Note that (3.6), (3.7), in contrast to (2.3), do not involve the curvature.

It is possible to extend the above analysis to the case when the initial, rather than the final geometry is specified. Let the inner wall of the uncharged cylinder be:

$$r = a(1 - \epsilon) + \delta u(\theta)$$

where

$$\epsilon \quad \frac{(1 - \nu^2)pa}{hE} .$$

Then, the change in curvature is:

$$\frac{-\epsilon}{R} + \frac{\delta}{R^2}[\ddot{u} + u - \ddot{v} - v]$$

and hence

$$\frac{h^3 E}{12(1 - \nu^2)R^2}[\ddot{u} + u - \ddot{v} - v] = -pRv$$

or, equivalently,

$$\ddot{v} + \left(1 - \frac{3p}{p^*}\right)v = \ddot{u} + u \ . \tag{3.8}$$

Equation (3.8) can be solved using Fourier series. Let

$$u(\theta) = \sum_{k=2}^{\infty} (\alpha_k \sin k\theta + \beta_k \cos k\theta) \ .$$

Then

$$v(\theta) = \sum_{k=2}^{\infty} \frac{k^2 - 1}{(k^2 - 1 + 3p/p^*)} (\alpha_k \sin k\theta + \beta_k \cos k\theta) \ .$$

Thus,

$$\sigma_{max} \approx \frac{pR}{h} + \frac{6\delta ap}{h^2} \times$$

$$\max\left\{\left| \sum_{k=2}^{\infty} \frac{k^2 - 1}{(k^2 - 1 + 3p/p^*)} (\alpha_k \sin k\theta + \beta_k \cos k\theta)\right|\right\} .$$

For the special case of an ellipse, we obtain:

$$\sigma_{max} \approx \frac{pa}{h}\left(1 + \frac{6\delta}{h}\max|u| \Big/ \left(1 + \frac{p}{p^*}\right)\right)$$

which is precisely (1.5). However if the shape is arbitrary, (1.5) will almost certainly be an underestimate.

A bound for v can be obtained in principle by noting that

$$v(\theta) = \int_0^{2\pi} g|\theta - \xi||\ddot{u}(\xi + u(\xi)|\, d\xi$$

where

$$G(\phi) = \frac{\cos(\beta\pi - \beta\phi)}{\beta \sin \beta\pi} - \frac{1}{2\pi\beta^2} - \frac{1}{\pi}\frac{\cos \phi}{\beta^2 - 1}$$

and

$$\beta^2 = 1 - \frac{3p}{p^*}.$$

Thus,

$$|v(\theta)| \leq C \max|\ddot{u} + u|$$

and

$$C = \max_{\theta} \int_0^{2\pi} |G(|\theta - \xi|)|\, d\xi .$$

An estimate for C was not obtained during the study group, although it is possible to show that

$$C \leq \frac{1}{3p/p^* - 1}, \qquad p > p^*/3 .$$

The above analysis shows that we need to measure:

(i) For charged cylinders - the maximum deviation in the radius

OR

(ii) For uncharged cylinders - the maximum deviation in the curvature.

4. THICK-WALLED VESSELS

Consider a thick shell, and choose cylindrical polars (r,θ,z). We assume that the strain components are independent of z, and further, that

$$\epsilon_{rz} = \epsilon_{\theta z} = \epsilon_{zz} = 0 \; ; \tag{4.1}$$

i.e., we assume plain strain.

In this situation, the stresses σ_{rr}, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ are all derivable from a potential function Φ, which is biharmonic in the annular region of the tube, and which yields

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} , \tag{4.2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} , \tag{4.3}$$

$$\sigma_{r\theta} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial r}\right) , \tag{4.4}$$

and

$$\sigma_{rz} = 0, \quad \sigma_{\theta z} = 0 , \quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) , \tag{4.5}$$

where ν is Poisson's ratio for the material of the tube (about 0.3 for steel).

Now consider a *charged* cylinder whose cross-section is defined by

$$r_1(\theta) \leq r \leq r_2(\theta) , \tag{4.6}$$

with

$$r_1(\theta) = a + \delta \cos 2\theta ,$$

$$r_2(\theta) = b + q\delta \cos 2\theta ,$$

where $(\delta/a) << 1$.

For a cylinder subject to internal pressure p at the inner boundary and zero pressure at the outer, we have the boundary conditions

$$\sigma_{nn} = -p \; ; \quad \sigma_{nt} = 0 \quad \text{at} \quad r = r_1(\theta) , \tag{4.7}$$

and

$$\sigma_{nn} = 0\ ; \quad \sigma_{nt} = 0 \quad \text{at} \quad r = r_2(\theta)\ , \tag{4.8}$$

where subscripts n and t denote stress components normal and tangential to the boundary respectively.

If we assume an elastic potential of the form

$$\Phi(r,\theta) = A \ln\left(\frac{r}{a}\right) + B\left(\frac{r}{a}\right)^2 + (\delta \cos 2\theta)$$

$$\left\{C\left(\frac{r}{a}\right)^4 + D\left(\frac{r}{a}\right)^2 + E + F\left(\frac{a}{r}\right)^2\right\}, \tag{4.9}$$

where A, B, ..., are constants; we obtain, on substituting into (4.2)-(4.4) and applying (4.7), (4.8) to leading order in δ, a set of equations determining A, B, ..., etc.

These then give

$$A = -\frac{pa^2b^2}{b^2 - a^2}\ ;$$

$$B = \frac{pa^4}{2(b^2 - a^2)}\ ;$$

$$C = -\frac{pba^5(qa + b)}{(b^2 - a^2)^3}\ ;$$

$$D = -\frac{pa^3b\{qa(2a^2 + b^2) + b(a^2 + 2b^2)\}}{(b^2 - a^2)^3}\ ;$$

$$E = -\frac{pab\{qa^3(a^2 + 2b^2) + b^3(2a^2 + b^2)\}}{(b^2 - a^2)^3}\ ;$$

$$F = \frac{pab^3(qa^3 + b^3)}{(b^2 - a^2)^3}\ .$$

In particular, the hoop stess σ_{tt}, regarded as being of most interest in this analysis, is given by

$$\sigma_{tt} = p\left\{\frac{a^2(b^2 + r^2)}{(b^2 - a^2)r^2} + \frac{6a^3b^3}{(b^2 - a^2)^3 r^4}(\delta \cos 2\theta)(b^3 + qa^3)\right.$$

$$+ \frac{2ab}{(b^2 - a^2)^3}(\delta \cos 2\theta)(qa(2a^2 + b^2 - 6r^2) \tag{4.10}$$

$$\left. + b(a^2 + 2b^2 - 6r^2))\right\} + 0(\delta^2) .$$

Hence, when $q = 1$ (uniform wall thickness)

$$\sigma_{max} = \frac{p(b^2 + a^2)}{b^2 - a^2} + \frac{4pb\delta(2a^3 + b^3 + 3a^2b)}{a(b^2 - a^2)^2} . \tag{4.11}$$

Although the correction due to non-circularity can be large, (4.11) is consistent with (1.3) and (3.6). Furthermore, a preliminary investigation indicates that the terms neglected are $O(\delta^2/h^3)$ which suggests that the analysis above is valid if δ/h is small.

It is worthwhile rewriting (4.11) as

$$\sigma_{max} = \frac{pR}{h}\left(1 + \left(\frac{h}{2R}\right)^2\right) + \frac{6\delta pa}{h^2}\left(1 + \frac{h}{R}\right) + O(\delta p)$$

as this shows that the estimate (3.6) (when the charged cylinder is elliptical) has a *relative* error of approximately

$$\frac{(h/2R)^2 + 6(\delta/R)}{1 + 6\delta/R} .$$

Thus (3.6) appears to be adequate for most practical purposes. Note however that if h/R is not small, stresses such as the normal stress need to be taken into account when designing pressure vessels.

The above analysis can be generalized to the case when the shape of the *charged* cylinder is more complex. However the analysis is valid only if the final shape of the cylinder is known.

5. NON-UNIFORM WALL THICKNESS

The effect of non-uniform wall thickness did not receive much attention due to time constraints. However, a solution for the cross section in figure 2 was found in the literature [5]. The maximum tangential stress is

$$\sigma_{max} = p\left[\frac{2b^2(b^2 + a^2 - 2ae - e^2)}{(b^2 + a^2)(b^2 - a^2 - 2ae - e^2)} - 1\right]$$

when the external pressure of the cylinder is p. When e = 0, this of course reduces to the standard formula for a cross section $a \leq r \leq b$ of concentric circles. Thin shell theory, as employed in section 3, yields the approximation

$$\sigma_{max} \approx p\left[\frac{b + a - e}{2(b - a - e)}\right]$$

which shows that the maximum stress is nearly the same as that of two concentric circles of radii r = a and r = b - e.

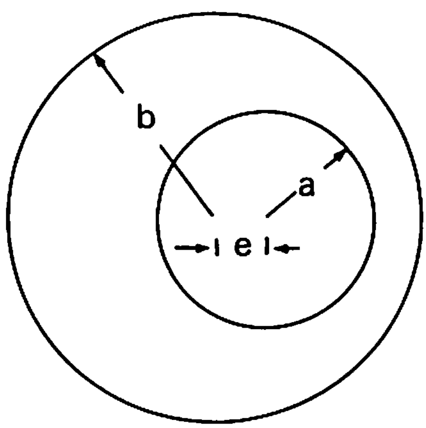

Figure 2: Eccentric Bore.

6. DISCUSSION

The question of formulating a yield criterion was not examined. However, it is clear that when h/R is not small, stresses other than the tangential stresses need to be taken into account. Given that h/R *is small* there are two cases which can be considered.

a. *Charged Cylinders.* In this case, the estimate (1.3) is appropriate. All that has to be measured is the maximal radial deviation from circularity.

b. *Uncharged (empty) Cylinders.* In this case, the maximum deviation in *curvature* from circularity must be measured. In practice this will be quite difficult unless a simple parameterization of the out-of-roundness is assumed. Given that the shape is elliptical, the estimate (1.5) is appropriate. The basis of (1.6) is unclear although the factor of 4 in the numerator is possibly a safety factor. However, (1.6) does not give an upper bound for an arbitrary shape.

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