THIN-WALLED BEAM OPTIMISATION

1. Introduction

General Motors-Holden's Automotive Ltd (GMH) provided the Study Group with a problem involving the optimisation of cross-sectional shapes of various hollow beams which occur in car body structures such that the weight can be reduced without sacrificing style, strength and stability requirements.

In the design of the overall vehicle structure, preliminary analyses are made for vibrations and strength. At this early stage, the beams are represented as lines with undefined cross section, merely points which should have appropriate moments of inertia in horizontal and vertical directions relative to the ground and a torsional constant. The next phase is to flesh out these beams, firstly taking any styling into consideration and then to meet the target values of the preliminary design. Then a final detailed analysis of the whole structure, including finite cross sections, is performed using a comprehensive finite element analysis. Beams at this stage are considered three dimensionally near their connections and as line elements in between. Several iterations in these phases could well be necessary. The participants at the Study Group decided as a first step to look at the class of hollow thin-walled beams (Gjelsvik, 1981; Murray, 1984; Vlazov, 1961) of closed cross section. The walls are sufficiently thin to neglect the effects of warping which can become important in correspondingly thin open section beams. With classical thin-walled theory being applicable, it would be natural to expect that such optimisation problems would have been well studied. A modest data base search of the open literature yielded only one relevant publication (DeVries *et al.*, 1986).

The overall question which needs to be answered is whether or not optimisation would be cost effective to GMH in improving current beam designs. If so, what is the best way of doing it? Here, a beginning is made on the second of these questions.

There are essentially two basic approaches to solving our optimisation problem. One is to represent the unknown cross-sectional curve by some parameters, such as coordinates of splined segment joints, and then optimally select these by one of the standard numerical algorithms available. The other is to base the optimisation on the calculus of variations, a method well suited here. Both procedures are examined, with more emphasis given to variational calculus.

2. Problem definition

Figures 1(a) and 1(b) show cross sections of two of a family of single or multi-celled, hollow and thin-walled beams which may occur in the frame of a car body. The wall thicknesses are denoted by t, arcs by S, arc lengths by L and enclosed areas by A. It is required to find the dotted curve S and its thickness t given the solid curve S_0 and its thickness t_0 such that predetermined section properties I_{xx}, I_{yy} and J are met and the weight of the beam is minimised. There will be some constraints on allowable thicknesses and on the extent of the available region for S defined by a bounding envelope E. The junction between S and S_0 has been idealised, in that a real junction will have short flanges usually outwardly normal to S–S_0 and spot welded along the length of the beam. This refinement can be included easily in a future analysis.

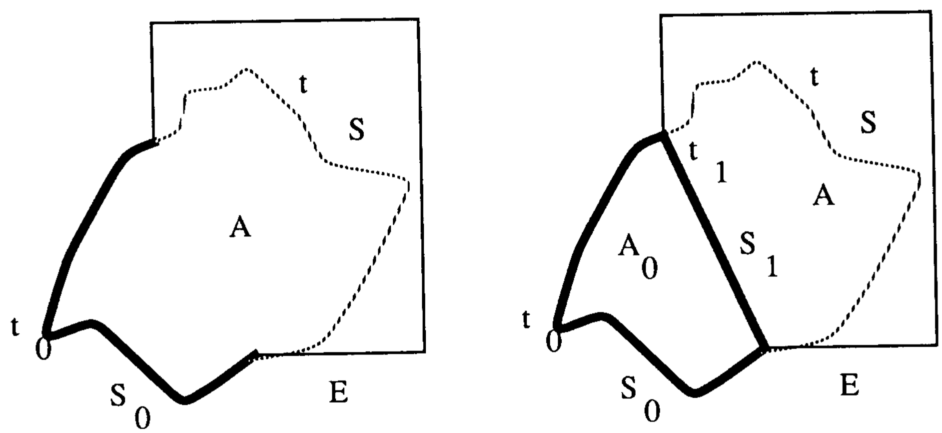

Figure 1: Cross sections of hollow, thin-walled beams

Because the cross-sectional shape of the beams to be designed is constant along their length, their weight, which for a given homogeneous material is proportional to volume, is then proportional to sectional area of the tube material. With the further practical fabrication constraints of making continuous sections of constant thickness, weight in such sections can be assumed proportional to lengths L_0, for S_0, and L, for S. Contours and lengths are based on mean paths within wall cross sections.

Now define the requirements mathematically:

The axes x,y are principal axes located at sectional centroids. The terminology for the single cell beam is used.

The position of the centroid is defined by

$$0 = t_0 \int_{S_0} x ds + t \int_S x ds \qquad (1)$$

$$0 = t_0 \int_{S_0} y ds + t \int_S y ds \qquad (2)$$

with $ds = (dx^2 + dy^2)^{1/2}$.

Arc lengths and enclosed areas are

$$L_0 = \int_{S_0} ds \qquad (3)$$

$$L = \int_S ds \qquad (4)$$

$$A = \int_{S_0} y dx + \int_S y dx \qquad (5)$$

The torsional constant J is the Saint Venant torsion constant which expresses the effects of geometry in the relationship between cross-sectional twisting moment and angle of twist. For a single cell it is (Gjelsvik, 1981; Murray, 1984)

$$J = \frac{4A^2}{\int_{S_0+S} \frac{ds}{t}}$$

which becomes

$$J = \frac{4A^2}{\left(\frac{L_0}{t_0} + \frac{L}{t}\right)} \qquad (6)$$

For a multi-cell, the extension of the definitions above is simple except for J. However, this is well known and may be found in texts such as Gjelsvik (1981) and Murray (1984). For our purposes it is sufficient to present the results for a two-celled beam as in Figure 1(b):

$$J = 4\left\{\frac{L_0 t t_1 A^2 + L t_1 t_0 {A_0}^2 + L_1 t_0 t (A + A_0)^2}{L_0 L t_1 + L L_1 t_0 + L_1 L_0 t}\right\} \qquad (7)$$

With L_0, L_1, t_0, t_1 and A_0 known and L, t assumed, then with c_1, c_2 and c_3 constants, defined from equation (7),

$$J = c_1 + c_2 A + c_3 A^2 \tag{8}$$

For a greater number of cells the quadratic form in A still holds. Whether or not a beam is multi-celled is predetermined by other criteria such as localised buckling.

The principal moments of inertia are given by

$$I_{xx} = t_0 \int_{S_0} y^2 ds + t \int_S y^2 ds \tag{9}$$

$$I_{yy} = t_0 \int_{S_0} x^2 ds + t \int_S x^2 ds \tag{10}$$

$$I_{xy} = t_0 \int_{S_0} xy ds + t \int_S xy ds = 0 \tag{11}$$

Practical restrictions limit the thickness to a lower bound t_L and an upper bound t_U

$$t_L \leq t \leq t_U \tag{12}$$

The nonlinear weight function to be minimised is

$$W = t_0 L_0 + tL \tag{13}$$

which is equivalent to a minimised L when t is given.

Practically, it is recognised that it may be difficult to achieve all three targets set on I_{xx}, I_{yy} and J. The value of I_{yy} is considered to be the least critical for vibrational design and may be allowed to float. Furthermore, it may be adequate to have axes non-principal with $I_{xy} \neq 0$.

3. Direct numerical optimisation

The conventional way of tackling our nonlinear optimisation problem is to adopt a curve shape S and a particular t and vary them systematically until a deemed minimum W is found which does not violate the constraints. The curve is normally assumed to be composed of a number of splined segments, linear, quadratic or cubic. Some simple calculations for linear segments will now be performed.

Given two fixed points with their cartesian coordinates, $p_0(x_0, y_0)$ and $p_{n+1}(x_{n+1}, y_{n+1})$, join them with $n+1$ straight-line segments numbered sequentially. The location of nodes at points $p_1(x_1, y_1)$, ..., $p_n(x_n, y_n)$ then define the segments representing the unknown boundary S with a total of $2n$ unknown coordinates and 4 known. Consider now the j^{th} segment lying between $p_{j-1}(x_{j-1}, y_{j-1})$ and $p_j(x_j, y_j)$ defined by

$$y = a_j x + b_j \tag{14}$$

with

$$a_j = \frac{(y_j - y_{j-1})}{(x_j - x_{j-1})}, \quad b_j = \frac{(x_j y_{j-1} - x_{j-1} y_j)}{(x_j - x_{j-1})}$$

The integrals defining the sectional properties may now be calculated by summation over the individual segments. The position of the centroid $(\overline{x}, \overline{y})$, for principal axes x, y derived from an arbitrary set of axes x', y', then follows from equations (1) and (2) as

$$\overline{x} = \frac{t_0 \int_{S_0} x' ds + \frac{1}{2} t \sum_{j=1}^{n} (1 + a_j^2)^{\frac{1}{2}} (x_j'^2 - x_{j-1}'^2)}{t_0 L_0 + tL} \tag{15}$$

$$\overline{y} = \frac{t_0 \int_{S_0} y' ds + \frac{1}{2} t \sum_{j=1}^{n} (1 + a_j^{-2})^{\frac{1}{2}} (y_j'^2 - y_{j-1}'^2)}{t_0 L_0 + tL} \tag{16}$$

where $x_j' - x_{j-1}' = x_j - x_{j-1}$ and $y_j' - y_{j-1}' = y_j - y_{j-1}$ are required for a_j above and in

$$L = \sum_{j=1}^{n} \left[(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 \right]^{\frac{1}{2}} \tag{17}$$

For a single cell

$$A = \int_{S_0} y dx + \sum_{j=1}^{n} [\frac{1}{2} a_j (x_j^2 - x_{j-1}^2) + b_j (x_j - x_{j-1})] \tag{18}$$

and J remains as equation (6). The moments of inertia are

$$I_{xx} = t_0 \int_{S_0} y^2 ds + \frac{t}{3} \sum_{j=1}^{n} (1 + a_j^{-2})^{\frac{1}{2}} (y_j^3 - y_{j-1}^3) \tag{19}$$

$$I_{yy} = t_0 \int_{S_0} x^2 ds + \frac{t}{3} \sum_{j=1}^{n} (1 + a_j^2)^{\frac{1}{2}} (x_j^3 - x_{j-1}^3) \tag{20}$$

$$\begin{aligned} I_{xy} &= t_0 \int_{S_0} xy ds + \frac{t}{3} \sum_{j=1}^{n} (1 + a_j^2)^{\frac{1}{2}} [a_j (x_j^3 - x_{j-1}^3) + \frac{3}{2} b_j (x_j^2 - x_{j-1}^2)] \\ &= 0 \end{aligned} \tag{21}$$

The objective function W remains as equation (13). For a fixed t and no envelope E the conditions for extrema are

$$\frac{\partial W}{\partial x_j} = \frac{\partial W}{\partial y_j} = 0, \quad j = 1, \ldots, n$$

and from these the global minimum is sought. With the envelope E included and the thickness t variable but bounded as in equation (12), the minimum W could well have some point p_j on E and $t = t_L$ with $\partial W/\partial t \neq 0$.

For quadratic and cubic spline segments it is not difficult to establish equations analogous to the linear case above. Then integrations provide logarithmic functions and elliptic integrals respectively.

There are well established algorithms (Gill *et al.*, 1981) for solving our optimisation problem. These are not discussed here; instead, a special case is considered with a small number of linear segments without envelope restraint. This forms a basis for comparison with the variational calculus approach of the next section. The special case is a non trivial one not appearing in standard structural analysis texts. It is the determination of the contour S, $y \geq 0$, of minimum length L between two points, each on the line $y = 0$ at distance $2b$ apart, for a given I_{xx} about $y = 0$. This problem is identical to solving its reciprocal: given L find S which maximises I_{xx}. This is the problem which is now examined.

b	.125	.250	.375	.500	.625	.750	.875
Shape							
1/2 Rectangle	.6380	.5625	.4557	.3333	.2109	.1042	.0286
1/2 Diamond	.6563	.6250	.5729	.5000	.4063	.2917	.1563
1/2 Hexagon	.6585	.6328	.5876	.5208	.4307	.3151	.1722
1/2 Octagon	.6596	.6361	.5933	.5286	.4386	.3232	.1772
sn Elliptic Function	.6611	.6406	.6009	.5385	.4504	.3333	.1841

Table 1. I_{xx}/t for shapes of Figures 2(a,b,c,d,e) with $L = 2$.

Figures 2 (a, b, c, d) display linear segment approximations to the true curve calculated in Section 4 (an elliptic *sn* function) of Figure 2(e). Each has a length $L = 2$ units. Table 1 shows values of I_{xx}/t for variations of b. The symmetrical shapes of Figures 2(a,b,c,d) are first assumed and I_{xx}/t calculated for various positions of the moveable nodes. Although the height of each figure changes with base width, the semi-rectangle and -diamond are fixed once b is set. The semi-hexagon has one degree of freedom, for a given b, to preserve its top surface level at $y = y_1$ for the node p_1 shown and determined from $\partial I_{xx}/\partial y_1 = 0$. The semi-octagon has two degrees of freedom, x_1 and y_1 for the node p_1, determined

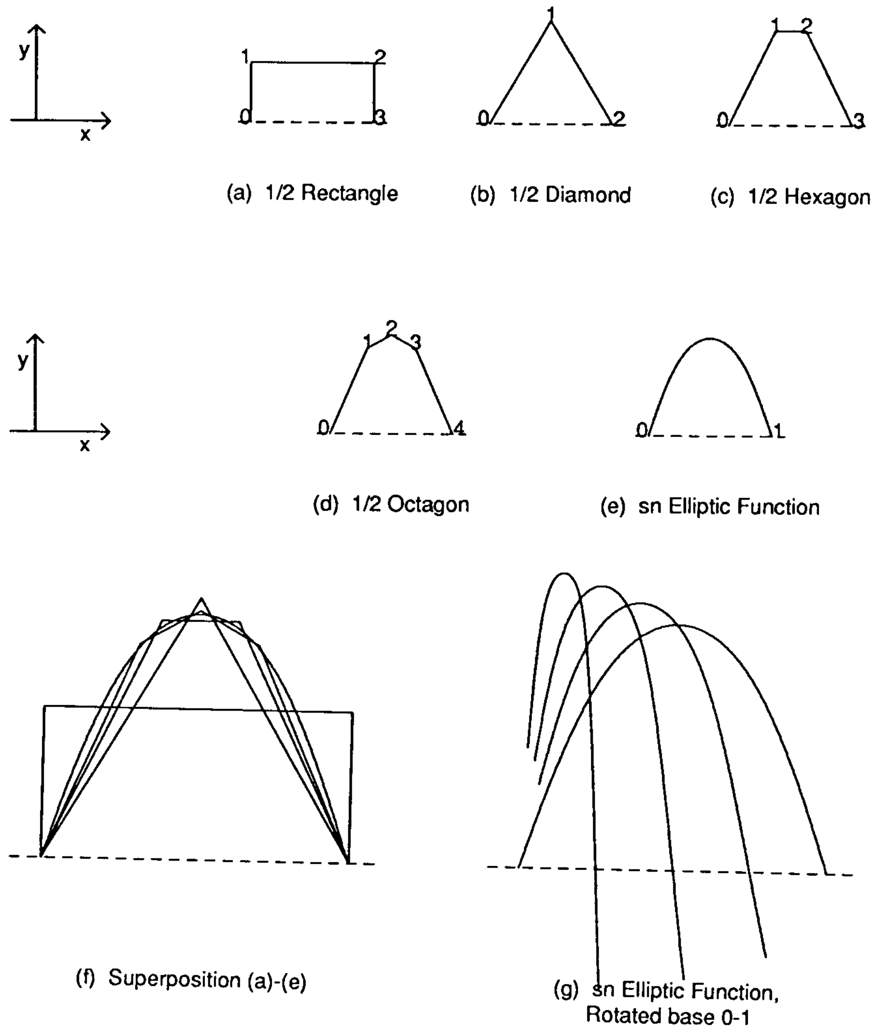

Figure 2: Shapes leading to optimum sn elliptic functions: maximised moment of inertia I_{xx}, arc length $L = 2$, base points separation $2b = 1$

from $\partial I_{xx}/\partial x_1 = \partial I_{xx}/\partial y_1 = 0$. From the numerical results of Table 1, and the superposition of correctly scaled Figures 2(a – e) in Figure 2(f), it would appear that linear segment representation for S will approach the correct smooth curve limit as the number of degrees of freedom is increased.

4. Optimisation by calculus of variations

(a) General formulation without envelope constraint

The objective function W containing L, its target constraints, $I_{xx}, I_{xy}, I_{yy}, J$ and centroid coordinates $\bar{x}, \bar{y}$, are all expressed in terms of line integrals. This immediately suggests that the optimisation can be done by a variational calculus approach for these quantities, with constant values of t discretely varied within the bounds of equation (12) until minimum W is found. Here the presence of E is ignored.

With $y' = dy/dx$ and Lagrangian multipliers $\lambda_{0,\ldots,6}$, consider a function

$$F(x, y, y') = (\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 x^2 + \lambda_4 xy + \lambda_5 y^2)(1 + y'^2)^{\frac{1}{2}} + \lambda_6 y \quad (22)$$

and a functional containing it

$$G = \int_{x_0}^{x_1} F(x, y, y')dx \quad (23)$$

The curve $y = f(x)$ between points $p_0(x_0, y_0)$ and $p_1(x_1, y_1)$ is to be found such that G has a global (absolute) extremum, either minimum or maximum. The association of the λ_i with section quantities is clearly

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \ : \ (L, \bar{x}, \bar{y}, I_{yy}, I_{xy}, I_{xx}, A) \quad (24)$$

This form of F is the well studied isoperimetric case (Gelfand & Fomin, 1963) where there is minimax reciprocity. That is, if $\lambda_0 = 1$, the minimum of G minimises L subject to six additional requirements of section quantities. This is the same as maximising any other one of the section quantities, say I_{xx}, replacing an I_{xx} requirement by one with L. Before discussing the use of area A instead of torsion constant J, the requirements (Gelfand & Fomin, 1963) for finding an extremum of G are briefly outlined.

From the first variation of G there is the essential requirement of the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad (25)$$

from which simplified forms devolve for $F = F(y, y')$,

$$F - y'\frac{\partial F}{\partial y'} = \text{constant} \quad (26)$$

and, if need be, for $F = F(x, y')$,

$$\frac{\partial F}{\partial y'} = \text{constant} \quad (27)$$

Equations (25) or (26, 27) produce differential equations of second or first order respectively but both involving two arbitrary constants d_1, d_2 in their solution $y = f(x, d_1, d_2)$. The end points p_0 and p_1 then provide two conditions

$y_0 = f(x_0, d_1, d_2)$ and $y_1 = f(x_1, d_1, d_2)$. Often the solutions for d_1 and d_2 are not unique and correspond to stationary values of G. These may be global and minimum, maximum or neither, or multiple extrema and any of the extrema may be weak (necessary continuity of y and y') or strong (necessary continuity of y only). (Weak and strong used here follow the calculus of variations tradition which is opposite to that generally used elsewhere in mathematics). For uniqueness of d_1 and d_2 additional conditions are imposed arising from second variations of G and the analysis of Weierstrass (Gelfand & Fomin, 1963). Although there are established necessary conditions and sufficient conditions for weak local extrema, and strong local and global, extrema, there do not appear to be available yet both necessary *and* sufficient conditions for either local or global extrema. Here, two sufficient conditions will be used which provide a strong global extremum. The first is the strengthened Weierstrass condition for a minimum

$$\frac{\partial^2 F}{\partial y'^2} > 0 \,, \qquad -\infty < y' < \infty \tag{28}$$

assuming $\partial^3 F/\partial y'^3$ exists. (For a maximum the sign of the inequality is reversed.) The second is the Jacobi condition that x solutions (conjugate points) of

$$\frac{\partial y}{\partial d_1} / \frac{\partial y}{\partial d_2} = \left(\frac{\partial y}{\partial d_1} / \frac{\partial y}{\partial d_2} \right)_{x=x_0} \tag{29}$$

must lie outside the open interval (x_0, x_1). If there are conjugate points inside (x_0, x_1) it means that either there is no extremum or that there is an extremum plus a stationary but non-extremum G. In practice, the Weierstrass condition (28) is used explicitly and the Jacobi condition (29) is more conveniently used implicitly by checking numerically for the most extreme G if multiple solutions are found for d_1 and d_2.

The nonlinear second order differential equation arising from (22) and (25) is

$$(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 x^2 + \lambda_4 xy + \lambda_5 y^2) y'' - (\lambda_2 + \lambda_4 x + 2\lambda_5 y)(1 + y'^2)$$
$$-\lambda_6 (1 + y'^2)^{\frac{3}{2}} + (\lambda_1 + 2\lambda_3 x + \lambda_4 y) y' (1 + y'^2)^2 = 0 \tag{30}$$

with Weierstrass condition

$$\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 x^2 + \lambda_4 xy + \lambda_5 y^2 \;>\; 0 \tag{31}$$

Likewise from (26) and (28) for $F(y, y')$ there is the first order differential equation and Weierstrass condition

$$(\lambda_0 + \lambda_2 y + \lambda_5 y^2)(1 + y'^2)^{-\frac{1}{2}} + \lambda_6 y = \text{constant} \tag{32}$$

$$\lambda_0 + \lambda_2 y + \lambda_5 y^2 \;>\; 0 \tag{33}$$

If J of equation (6) were added to F as the integral form of $\lambda[J-4A^2/(t_0L_0+tL)]$ the first variation would produce $4A^2/(t_0L_0+tL)\lambda[t/(t_0L_0+tL)\delta L-(2/A)\delta A]$. The coefficients of δL and δA would then be absorbed into λ_0 and λ_6 respectively. A similar process is required when using the multi-cell equation (7). However, now it is necessary to use a constraint equation involving J, A and L, strongly suggesting that, when J is given and L is sought, a numerical iterative process is required using all the equations in finding an extremum G.

It is clear that with constant t a minimum W corresponds to a minimum G with $\lambda_0 = 1$. To gain some insight into the complete problem, it is useful to consider some special cases where some of $\lambda_i = 0$.

(b) Maximise I_{xx} given L

Initially the case numerically studied in Section 3 with piecewise linear segments will be re-examined to obtain the curve of Figure 2(f). Variationally, equations (32) and (33), with $\lambda_2 = \lambda_6 = 0$ and $\lambda_5 = 1$, become

$$y^2 + \lambda_0 = -d_1(1+y'^2)^{\frac{1}{2}} \tag{34}$$

$$y^2 + \lambda_0 < 0 \tag{35}$$

These require that λ_0 is negative, say $\lambda_0 = -e$, e positive. Then

$$d_1 > 0 \quad \text{and} \quad y^2 - e < 0$$

$$d_1^2\left(\frac{dy}{dx}\right)^2 = (e + d_1 - y^2)(e - d_1 - y^2) \tag{36}$$

requiring that

$$e > d_1 \quad \text{and} \quad e - d_1 > y^2$$

With positive square root, equation (36) yields

$$x + d_2 = d_1 \int_0^y [(e + d_1 - u^2)(e - d_1 - u^2)]^{-\frac{1}{2}} du \tag{37}$$

Setting the origin of axes x, y at p_0, $d_2 = 0$ and the integral expression (37) may be written (Byrd & Friedman, 1971)

$$x = qF\left(\frac{(1-m)}{2qm^{1/2}} y \mid m\right) \tag{38}$$

$F(\,|\,)$ is the elliptic integral of the first kind with modulus

$$m = \frac{(e - d_1)}{(e + d_1)} \quad \text{and} \quad q = \frac{d_1}{(e + d_1)^{1/2}}$$

(Because of widespread usage, the notations $F(\ |\ )$ and $F(x, y, y')$ have not been altered but should not cause confusion.) Equation (38) can be inverted in terms of the $sn(\ |\ )$ elliptic function (Byrd & Friedman, 1971):

$$y = \frac{2qm^{1/2}}{(1-m)}\mathrm{sn}(\frac{x}{q} \mid m) \tag{39}$$

The $sn(u|m)$ function, containing $sin(u) = sn(u|0)$, is periodic with period $4K(m)$, $K(m)$ being the complete elliptic integral of the first kind. Equation (37) for positive gradient dy/dx, corresponds to the first quarter segment of sn. A negative gradient analogue of equation (37) would also produce the same sn function but for the second quarter segment. The constant q can now be evaluated at half span $x = b$ or at full span $x = 2b$ at p_1 as

$$q = \frac{b}{K(m)} \tag{40}$$

To find m use

$$\begin{aligned} L &= 2\int_0^b (1+y'^2)^{\frac{1}{2}}dx \\ &= -\frac{2}{d_1}\int_0^b (y^2 - e)dx \end{aligned}$$

After substitution of y from equation (39) and recognising an identity involving $E(m)$, the complete elliptic integral of the second kind, there is

$$L = \frac{2b}{(1-m)}[2\frac{E(m)}{K(m)} - (1-m)] \tag{41}$$

Because this problem is equivalent to minimising L given I_{xx}, it is expected that a somewhat similar expression may be found for I_{xx}. Indeed,

$$\begin{aligned} I_{xx} &= 2\int_0^b y^2(1+y'^2)^{\frac{1}{2}}dx \\ &= -\frac{2}{d_1}\int_0^b y^2(y^2 - e)dx \end{aligned}$$

producing

$$I_{xx} = \frac{4bq^2}{3(1-m)^3}[(1+m)\frac{E(m)}{K(m)} - (1-m)] \tag{42}$$

Thus m can be determined numerically from (41), given L and b, thence q from equation (40) and from (42). The I_{xx} values which appear in Table 1.

Efficient algorithms for calculating $K(m)$, $E(m)$ and sn were obtained from Abramowitz & Stegun (1965). Only a single m was found in (x_0, x_1), giving assurance that the Jacobi condition is satisfied and a global minimum has been found. The only reference to this problem found by the Study Group moderator was a last century text of Greenhill (1894), where a heuristic treatment was given with deduction of the sn form (39) from the dynamics of a whirling chain. Equations (41) and (42) were not given and neither were any numerical results.

Figure 2(g) shows an optimum shape for a curve where p_0 and p_1 no longer lie on $y = 0$. It is also an sn curve. If L and the distance $2b$ between p_0 and p_1 are held constant, the modulus m will increase towards 1 as the angle of rotation, ϕ, of the line p_0–p_1 is increased. If b is replaced by $bcos(\phi)$ then m is still determined by equation (41) and the expression for y of equation (39) still holds. The curve S then becomes thinner and taller. At the critical value of $\phi = \pi/2$, p_0 is directly above p_1 and S is now a vertical straight line, rising $\frac{1}{2}(L + 2b)$ above p_1 then folded on itself, falling $\frac{1}{2}(L - 2b)$ to p_0 – a strong maximum situation.

If x and y were interchanged in all parts of the preceding solution, then the solution fits exactly the case of maximising I_{yy} given L and its reciprocal, minimising L given I_{yy}.

(c) Other cases

The following cases for particular λ_i within equation (22) are of some interest and can be analytically determined with known expressions:

- Given I_{xx} and $\bar{y}$, minimise L. This corresponds to equations (32) and (33) with $\lambda_6 = 0$. This is a straight forward extension of Section 4(b) and involves elliptic integrals, $F(\;|\;)$ and sn functions.
- Given A, minimise L or given L, maximise A. These are isoperimetric problems of antiquity with S being a circular arc. This means that the J, L case is also satisfied by circular arcs.
- Given I_{xx}, $\bar{y}$ and A, minimise L. This is the case defined by equations (32) and (33). The solutions involve elliptic integrals of the first and third kinds. If calculations lead to $\lambda_6 = 0$ identically, the required A is exactly that arising from targeted I_{xx} and $\bar{y}$ and the integrals of the third kind vanish. Conversely, if $\lambda_2 = \lambda_5 = 0$ arise then S is a circular arc due to A. This case provides the dominant requirement of the overall optimisation problem. A careful study of changes in I_{xx} and J will provide good insight into desired changes in S and *vice versa*.

- As indicated in Section 4(b), the I_{yy}, L case is equivalent to the I_{xx}, L case and there is similar equivalence for I_{yy}, $\bar{x}$, A, L.

The simultaneous presence of x and y terms in equations (30) and (31) is essentially that of the complete problem and no analytical solutions have been recognised.

(d) Solution of the general problem with envelope

Minimising W requires at least one iterative process for thickness t. From the discussions above, another would be required for satisfying J targets. In the presence of E, iterations will be needed if an unconstrained optimum S were to intersect E. A portion of the constrained S would then lie on E and the remainder would satisfy the geometrically unconstrained equations. A discussion of such free boundaries with obstacles may be found in Elliott & Ockendon (1982).

Presuming that the second order, nonlinear differential equation (30) could be solved in analytical form, with two arbitrary constants, the number of nonlinear equations to be solved for the most general case would be eight, with six Lagrangian multipliers, for each iterate of t. An alternative would be to reduce the nonlinear set to four equations involving the most important quantities, L, I_{xx}, and A, using the analysis discussed in Section 4(b,c) and iterating on $\bar{x}$, $\bar{y}$, I_{yy}, I_{xy} and t. The right mix of analytical and iterative schemes needs to be investigated.

A major point of importance is the difficulty in solving nonlinear two point boundary value problems. This point is often used in justifying the use of dynamic programming (Cooper & Cooper, 1981) which, in principle, is correctly applicable to calculus of variation problems with E. However, it has a 'curse of dimensions' wherein the number of independent variables that can be used is quickly limited by computer processing memory capacity. Usually this number of variables is low, at five to eight, which makes feasible application marginal in our case.

Apart from using numerical methods for solving the differential equation, a recent analytical approach introduced by Adomian (1988) is worthy of consideration. This method has solved many nonlinear equations such as (30) remarkably efficiently. In essence the method uses a generalised Taylor series. Continuation can be used when radii of convergence become too small.

5. Concluding remarks

Within the limits of the Study Group environment, and with some numerical calculations done following it, some progress has been made in developing approaches to the thin-walled, hollow beam optimisation.

By focussing on the variational calculus method, valuable information can be obtained for the major section quantities I_{xx} and J required in design. Still more analysis can be done in this area.

Of the approaches discussed – direct numerical optimisation, analytical variational calculus and dynamic programming – analytical variational calculus is favoured for gaining greater insight within the design process; dynamic programming is valued for its assurance of finding the global minimum weight in a well structured way; and the direct approach can be employed quickly using existing algorithm software.

Acknowledgements

The moderator is grateful for the assistance of Laurie Sparke of GMH for practical guidance and for the contributions of Study Group members, particularly those of Neville Fowkes, Ian Howells, Tony Miller, John Ockendon and Peter Watterson.

References

M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, 1965).

G. Adomian, *Nonlinear Stochastic Systems Theory and Applications to Physics* (Kluwer, 1988).

P.F. Byrd & M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer-Verlag, 1971).

L. Cooper & M.W. Cooper, *Introduction to Dynamic Programming* (Pergamon Press, 1981).

R.I. DeVries *et al.*, "Structural optimization of beam sections for minimum weight subject to inertial and crush strength constraints", *Proc 6th Int Conf Vehicle Struct Mech* (1986), 47-51.

C.M. Elliott & J.R. Ockendon, *Weak and Variational Methods for Moving Boundary Problems* (Pitman Advanced Publishing Program, Boston, 1982).

I.M. Gelfand & S.V. Fomin, *Calculus of Variations* (Prentice-Hall, 1963).

P.E. Gill, M. Murray & M.H. Wright, *Practical Optimization* (Academic Press, 1981).

A. Gjelsvik, *The Theory of Thin Walled Bars* (Wiley, 1981).]

A.G. Greenhill, *The Applications of Elliptic Functions* (MacMillan, 1894).

N.W. Murray, *Introduction to the Theory of Thin-Walled Structures* (Clarendon Press, 1984).

V.Z. Vlazov, *Thin-Walled Elastic Beams* (Israel Program for Scientific Translations, 1961).