MODELLING DISPERSION EFFECTS IN PAPER — EFFECTIVE THICKNESS ESTIMATES

The structural properties of paper and cardboard are important for the design of containers and of packaging. Amcor, a Melbourne-based producer of packaging, pulp and paper products, proposed the initial problem of explaining the large discrepancies which were sometimes observed between the nominal thickness, as determined by hard platen measurements, and the effective thickness which would account for the experimentally measured values of bending stiffness. The question was also raised as to whether or not ultrasonic waves could be used to measure directly, and on-line, the effective thickness.

Paper can be regarded as a fibre-composite material, with a pronounced anisotropy due to the preferential alignment of fibres in the machine direction as a consequence of the manufacturing process. The salient features of elastic (ultrasonic) wave propagation in an anisotropic plate will be presented, with an emphasis on addressing the questions raised above. In particular, it will be noted that the characterization of the first anti-symmetric (flexural) mode should provide a convenient on-line measurement of the flexural stiffness, which is the more relevant property for quality control and structural design than the effective thickness.

1. Introduction

Amcor is a Melbourne-based producer of pulp and paper products. With an annual operating revenue of about $3 billion, it is one of the largest such companies in the world. Paperboard, heavy paper, carton stock etc. are widely used as packaging materials and for containers. Of the various mechanical properties of such paper products, the two which are perhaps the most important for predicting their end-use behaviour are the compression strength and the bending stiffness (or flexural rigidity).

Stiffness can be measured non-destructively, whereas a proper measurement of strength is necessarily destructive. Consequently, there is considerable interest in establishing correlations between strength and stiffness, albeit for particular classes of paper products and within limited ranges of manufacturing-process parameters, and to use measurements of stiffness for quality control. This approach is particularly attractive if the stiffness measurements can be performed on-line.

A measurement of this type has been proposed relatively recently by researchers at the Institute of Paper Chemistry in the USA (Baum and Habeger, 1980). What is actually measured on-line is the speed of ultrasound in the machine direction (md), or in the cross direction (cd), at a relatively low (sonic) frequency, typically around 150 Hz. More precisely, the speed of the lowest symmetric plate-wave mode, S_o, is measured. The square of this wavespeed is equal to the Young's modulus E (md or cd, respectively) divided by the density ρ, according to a well-known relation attributed to Newton:

$$c^2(S_o) = E/\rho. \tag{1}$$

The density of paper products is traditionally specified in terms of the basis weight W, rather than the density, the relation between these two being simply

$$W = \rho t, \tag{2}$$

where t denotes the thickness. Thus, from the on-line measurement of the wavespeed $c(S_o)$, and knowledge of the basis weight W, which is a controllable process parameter, one can derive an on-line measurement of the extensional stiffness S_e,

$$S_e \equiv Et = Wc^2(S_o). \tag{3}$$

This sonic extensional stiffness has been shown to correlate well with destructively measured strength properties (Vahey, 1987).

However, from the viewpoint of predicting the end-use performance of paper-based containers, the bending stiffness S_b is the more structurally significant parameter, rather than the extensional stiffness. The conventional expression for S_b, which applies for cylindrical bending of an isotropic plate (Timoshenko and Woinowsky-Krieger, 1959), is

$$S_b = Et^2/12. \tag{4}$$

Assuming that this relation can also be applied to paper products, one can define an effective thickness t_{eff} as follows

$$t_{eff} = (12S_b/S_e)^{1/2}, \tag{5}$$

where S_b is interpreted as a direct measurement of the bending stiffness, determined for example from the deflection of an end-loaded cantilever, as in the Taber test (Koran and Kamdem, 1989). Thus S_b represents an independent measurement of stiffness from S_e, and t_{eff} can be interpreted as the value of thickness which must be used to predict correctly the measured bending stiffness S_b, starting from a measurement of the extensional stiffness S_e.

The problem submitted to the MISG by Amcor can now be stated as follows. Measurements undertaken by Amcor on various paper products showed in some cases asignificant discrepancy between the effective thickness t_{eff} and the nominal thickness t_{nom}, which is measured with hard platens. Since t_{eff} is the more significant parameter for assessing structural performance, as discussed above, this discrepancy has serious implications for on-line quality control. It was hoped that a mathematical model for the propagation of elastic waves in plate-like paper products may provide a method for determining, or estimating, t_{eff}. In particular, Amcor had noted a recent theoretical analysis by Habeger, Mann and Baum (1979), which showed a relatively sharp drop in the wavespeed as a function of frequency, and which associated this drop with the onset of significant deformation normal to the *md/cd* plane, i.e. with the onset of significant elastic displacement in the z-direction (*zd*), using the conventional choice of rectangular cartesian axes whereby the x-axis points in the machine direction (*md*), and the y-axis in the cross-direction (*cd*). It was hoped that a careful scrutiny of this analysis might indicate how the frequency at which the sharp drop occurs could be used to determine t_{eff}.

The main activity during the one-week MISG workshop was accordingly focussed on understanding the analysis of Habeger *et al.* (1979) for plate-waves in paper. The major contribution of the workshop, however, was to point out that a measure of the bending stiffness S_b can be obtained directly from on-line measurements of the wavespeed of the lowest anti-symmetric (or flexural) plate-wave mode A_o, rather than from similar measurements for the lowest symmetric (or extensional) mode S_o which has been used for on-line measurement of the extensional stiffness, as noted above. Thus, it is not necessary to determine an effective thickness, nor is it necessary to determine the frequency corresponding to the drop in wavespeed for the S_o mode: the structurally significant parameter S_b is best determined from another mode of plate vibration altogether, viz. A_o.

Although this statement seems eminently plausible, it does not appear to have been recognized or exploited in the industrial context of paper-making. Attention appears to have been focussed on the S_o mode, perhaps because the corresponding wavespeed is relatively independent of frequency for sufficiently low frequencies (lower than the frequency noted above at which a sharp drop occurs to another plateau value for wavespeed, characteristic of high frequencies). Thus, the low-frequency components of a wave packet in the S_o mode are non-dispersive, so that a wave packet constituted mainly from these low frequencies would propagate without distortion (or with negligible distortion). This may facilitate the practical measurement of wavespeed, which may account for the focus on the S_o mode. By contrast, the wavespeed of the A_o mode increases rapidly with frequency at low frequencies, levelling off to a plateau only at high frequencies. From a practical viewpoint, it is desirable to monitor the low-frequency wavespeed, because of the significant attenuation at high frequencies due to microstructural interactions and to viscoelastic effects. Habeger *et al.* (1979) report measurements of wavespeed for both the S_o and the A_o modes, showing reasonably good correlation between these measurements and the theoretical dispersion curves. However, it does not appear to have been appreciated that the wavespeed measurements for the A_o-mode could be used to define a *bending stiffness*, in the same manner that the wavespeed of the S_o-mode can be used to define the sonic extensional stiffness in Eq. (3). This MISG suggestion may therefore provide the basis for a new industry standard for on-line quality control. From a manufacturer's viewpoint, the desirable objective is to produce a specified bending stiffness at minimum basis weight and at the lowest possible cost. Reliable on-line monitoring of A_o-mode wavespeed would clearly assist in attaining this objective.

The remainder of this report is organized as follows. Section 2 presents the data provided by Amcor, including a brief discussion on the traditional measurement of bending stiffness. Section 3 summarizes the key concepts and results of the theory of plate-vibration modes for an isotropic material. This provides a valuable background for understanding the additional complications due to material anisotropy. The analysis of platewaves in an orthotropic material, due to Habeger *et al.* (1979), is also summarized in Section 3, focussing particularly on the S_o and A_o modes.

2. The bending stiffness of paperboard

Koran and Kamdem (1989) provide a useful survey of several procedures which are currently used for measuring the stiffness of paperboard, and of the correlations between these measurements. The two measurements of greatest interest for this report lead to (i) the static *extensional stiffness* $S_e = Et$, which is determined from a standard tensile test, using an Instron testing machine, and (ii) the static *bending stiffness*, or *flexural rigidity*, S_b, determined by using a cantilever specimen, of prescribed standard dimensions, clamped at one end and subjected to a transverse load at the other. This latter test is known as the Taber test. It records the load which must be applied (via a roller in contact with the paperboard at a fixed distance from the clamped end) to cause a prescribed deflection of 15°.

Table 1 shows the stiffness data obtained by Amcor for various paper products. It is noted that the *static* extensional stiffness, determined from a standard tensile test, correlates well with (but is consistently lower than) the *sonic* extensional stiffness, determined from the measured wavespeed of the S_o mode using Eq. (3). The Buchell stiffness is determined by using the same test configuration as for the Taber test described by Koran and Kammen (1989), but simply recording the applied load as a measure of bending stiffness, hence the data has units of mN, rather than $mN.m$ (or, more properly, $mPa.m^3$) which would be expected from Eq. (4) for the bending stiffness.

Hard-Platen Thickness (mm)	Effective Thickness (mm)	Discrepancy %	Static Extensional Stiffness (MPa.m)	Sonic Extensional Stiffness (MPa.m)	Buchell (Flexural) Stiffness (mN)
0.32	0.22	31	2.15	2.72	168
0.39	0.25	36	2.45	2.81	248
0.45	0.31	31	2.66	3.46	425
0.435	0.30	31	2.23	2.94	340
0.105	0.10	5	0.82	1.09	14
0.22	0.13	41	0.90	1.15	25
0.35	0.27	23	2.17	2.70	263
0.35	0.26	26	2.08	2.75	231
0.24	0.21	13	2.01	2.62	146
0.23	0.16	30	1.63	2.03	72
0.32	0.21	34	0.78	1.02	56
0.40	0.28	30	0.83	1.19	111
0.44	0.30	32	1.06	1.43	154
0.42	0.21	50	0.66	0.83	50
0.10	0.11	-10	0.33	0.46	6.58
0.22	0.14	36	0.35	0.45	11.75
0.36	0.29	19	0.59	0.81	85
0.35	0.27	23	0.59	0.87	72
0.29	0.22	24	0.57	0.77	46
0.23	0.18	22	0.47	0.70	24

Table 1: Amcor data for various paper products

The third column of Table 1 shows the discrepancy between *effective* and *nominal* thicknesses, which was mentioned in Section 1. It can be seen that the discrepancy varies significantly between various types of product, the larger values being associated with a larger surface roughness. However, it was soon established, in discussions with the Amcor representative, that the discrepancies could not be *attributed* solely to the surface roughness, which could only account for a discrepancy of less than 10% in the worst case. Instead, the MISG workshop identified the pronounced anisotropy of paper products as a possible contributing factor.

It is well recognized that machine-made paper products can be regarded as orthotropic materials, consisting of fibres (typically 2–3 mm long and approximately 0.02 mm in diameter) which are preferentially aligned in the machine direction (*md*), leading to a significantly higher Young's modulus in that direction than in the cross machine direction (*cd*) or the thickness direction (*zd*). The manufacturing process also leads to a distinctly layered structure, which is discernible even with the naked eye when tearing or rupturing cardboard and similar products, or when peeling off sticky tape from a paper substrate. Thus, the shear modulus in the x-z plane G_{xz} (*md*/*zd* plane) can be expected to be significantly smaller than the *md* Young's modulus, E_{md}, particularly in products of poorer quality.

A proper description of the elastic response of orthotropic materials requires the specification of nine independent stiffness-coefficients (or elastic constants), relating the stress components to the strain components. Mann, Baum and Habeger (1980) have presented a procedure for estimating all nine coefficients from ultrasonic (wavespeed) measurements, but with varying degrees of accuracy for the various coefficients. They comment in particular on low values of the *zd* stiffness coefficients obtained by this procedure for milk carton stock:

$$E_{md}/E_{zd} \approx 200, \qquad E_{md}/G_{xz} \approx 50.$$

For an isotropic material, these ratios would be equal to 1 and 2.6 respectively, assuming a typical value of 0.3 for the Poisson ratio. *This pronounced anisotropy casts doubts on the validity of classical expressions for cantilever deflection in estimating the bending stiffness of paper products.*

The crux of the problem is that the classical expression is based on a kinematic assumption which may not be appropriate in the presence of pronounced anisotropy. This kinematic assumption states that plane sections that are initially normal to the neutral axis (or the line of centroids) remain plane and normal to the neutral axis during deformation. It has long been recognized that the resulting theory ignores *the shear deformation* associated with flexure, and its contribution to the deflection. This is equivalent to assuming an infinite value for the shear modulus. The combination of zero deformation and infinite modulus nevertheless results in a finite *shear force*, acting normal to the neutral axis. This shear force has the chaacter of a reaction force, i.e. it is a force required so as to satisfy the conditions for equilibrium, but it has no deformation associated with it.

There have been several attempts to develop a refined theory of flexure taking into account transverse shear deformation, but it is not clear whether any of these would be entirely suitable for paper products. The problem is that although these theories eschew the classical kinematic assumption stated above, they still retain (implicitly or explicitly) a linear variation of the tensile stress σ_{xx} across the thickness,

$$\sigma_{xx}(x,z) \quad = \quad \frac{M(x)}{I} z, \tag{6}$$

where $M(x)$ denotes the bending moment at location x, and I the moment of inertia of the cross-section. For the case of cylindrical bending, which is an appropriate idealization for the Taber test, $M(x)$ is to be interpreted as the bending moment per unit length in the cross direction (cd), and

$$I \quad = \quad t^3/12. \tag{7}$$

This linear variation of σ_{xx} in Eq. (6) leads to the familiar parabolic variation for the shear stress σ_{xx}, through integration of the equilibrium equation

$$\sigma_{xx,x} + \sigma_{xx,z} \quad = \quad 0, \tag{8}$$

where a comma is used to indicate a partial derivative with respect to the variable(s) following the comma. However, Rogers and Pipkin (1971) have pointed out that for cases of pronounced anisotropy, an entirely different stress distribution may be more appropriate; in the limit of "inextensible fibres", i.e. for $E_{md}/G_{xz} \rightarrow \infty$, the deflection of a cantilever is entirely due to shear. The corresponding tensile stress can be written in the following form, instead of Eq. (6),

$$\sigma_{xx}(x,z) \quad = \quad \frac{M(x)}{t} \left\{ \delta(z - t/2) - \delta(z + t/2) \right\}. \tag{9}$$

The associated shear stress can again be determined from the equilibrium equation, Eq. (8): it has a constant value for $-t/2 < z < t/2$, but suffers step changes for $z = \pm t/2$, so as to satisfy the traction-free boundary condition on the free surfaces of the cantilever.

What would be desirable, but is currently not available, is a theory of flexure for anisotropic plates which exhibits a transition from Eq. (6) to Eq. (9) in the limit of high anisotropy.

It has not been possible within the time-frame of the MISG workshop to develop such a theory. However, as a suggestion for further work, a plausible approach would be to assume the following form for the thickness-variation of the tensile stress,

$$\sigma_{xx}(x,z) \quad = \quad \frac{M(x)}{t^2} \frac{2\eta^2 \sinh(\eta z/h)}{\eta \cosh \eta - \sinh \eta}, \tag{10}$$

where η can be regarded as a non-dimensional anisotropy index, so that for small η Eq. (10) reduces to Eq. (6), whereas for large η Eq. (10) reduces to Eq. (9).

This index should of course be expressed in terms of the elastic constants, but the appropriate relation is not immediately apparent. Valuable guidance can be obtained from the stress variation (across the thickness) associated with the exact solution for plate vibrations, as discussed further in Section 3.

To highlight the significance of accounting properly for the effect of anisotropy, it will be useful to record here the result based on Reissner's (1947) theory, which takes into account transverse shear deformation, for the end-deflection Δ of a cantilever of length l and thickness t, subjected to an end load P (per unit length in the cross direction),

$$\Delta = \Delta_b + \Delta_s\,, \tag{11a}$$

$$\Delta_b = 4\,(P/E_{md})\,(l/t)^3, \tag{11b}$$

$$\Delta_s = P/\left\{10(1+\nu_{xz})G_{xz}\right\}(l/t). \tag{11c}$$

In this simplified theory, the material properties are characterized by only three elastic constants: the axial Young's modulus, E_{md}; the axial shear modulus, G_{xz}; and the Poisson ratio ν_{xz}, which is the ratio of the transverse strain ε_{zz} to the axial strain ε_{xx} due to an axial normal stress σ_{xx}. For the Taber test, $l = 50\ mm$ (Koran and Kamdem, 1989), whereas a typical specimen thickness $t = 0.3\ mm$, so that $l/t = 167$. Consequently, the shear contribution Δ_s to the total deflection would be expected to be negligible compared with Δ_b, which represents the classical flexural contribution, in spite of the relatively large ratio of E_{md}/G_{xz} for paper products. The crucial point, however, is that the formulae in Eqs. (11b, c) for these two contributions are derived on the assumption of a linear variation for σ_{xx} in the thickness direction, as given by Eq. (6), rather than the more general expression in Eq. (9) which would depart significantly from a linear variation for large values of E_{md}/G_{xz}, and which could therefore lead to a quite different shear contribution relative to flexure. *The present practice in paper testing in effect ignores the possible contribution of shear deformation to the total deflection, and may therefore under-estimate the bending stiffness for cases where this shear contribution is significant. This in turn would lead to values of effective thickness which are significantly lower than the nominal (hard-platen) thickness.*

This point deserves further investigation. The relative importance of a shear contribution in the Taber test could be assessed experimentally by varying the cantilever length l, and plotting $\log(P/\Delta)$ versus $\log(l/t)$. If the shear contribution is indeed negligible, this plot should yield a straight line with a slope of -3; a significant deviation from this expected slope could be attributed to a significant shear contribution. Alternatively, a numerical experiment could be performed using a finite element package capable of large-deflection analysis and assuming the appropriate orthotropic elastic constants for various paper products.

3. Dispersion curves for plate waves

A plate of uniform thickness behaves as a waveguide for elastic waves: its dynamic response to a specified excitation can be expressed (in principle) as a superposition of its modes of free vibration. These plate-wave modes can be characterized by (i) the mode shape, describing the variation across the thickness of the elastic displacement $\mathbf{u}$, or of a stress component, and (ii) the dispersion curves, which give the relation between the angular frequency ω and the wavenumber k, or equivalently the relation between the phase velocity c and frequency ω, for each separate mode.

The general features of the wave modes for an orthotropic plate, as revealed by the analysis of Habeger *et al.* (1979), are quite similar to those for an isotropic plate. Consequently, it will be useful to review first the key concepts and results of the theory for isotropic plates, as presented in standard textbooks (Achenbach, 1973; Miklowitz, 1978), and then to point out the additional complications or features due to anisotropy. The aim is to characterize self-sustaining modes of vibration which satisfy the following equations of motion (in the absence of body forces),

$$\rho \ddot{\mathbf{u}} = \nabla . \sigma \tag{12a}$$

where ρ, $\ddot{\mathbf{u}}$, σ denote respectively the density, the elastic displacement and the stress tensor, using standard notation (see e.g. Achenbach, 1973, or Miklowitz, 1978), subject to traction-free boundary conditions on the plate faces,

$$\sigma_{zx} = \sigma_{zz} = 0, \quad \text{on } z = \pm h. \tag{12b}$$

The plate thickness is taken here to be $2h$, in accordance with the notation of Achenbach (1973) and Miklowitz (1978), instead of the previous notation t used in Section 2. By using the relation between stress and strain (Hooke's law), and between strain and displacement, the equation of motion (12a) can be expressed in terms of the elastic displacement $\ddot{\mathbf{u}}$ only,

$$\rho \ddot{\mathbf{u}} = (\lambda + \mu)\nabla(\nabla . \mathbf{u}) + \mu \nabla^2 \mathbf{u}, \tag{12c}$$

where λ, μ denote the Lamé constants, which are related to the more familiar elastic constants E, ν (the Young's modulus and Poisson ratio) as follows,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{12d}$$

A standard procedure for identifying the characteristic free-vibration modes is to consider a mathematical representation for $\mathbf{u}$ which corresponds to progressive waves of angular frequency ω propagating in the direction of the x-axis. By assuming plane-strain deformation, so that $u_y = 0$, and the remaining two components of $\mathbf{u}$ are independent of y, this representation takes the following form

$$u_x(x, z, t) = U(z)e^{i(kx-\omega t)}, \tag{13a}$$

$$u_z(x, z, t) = W(z)e^{i(kx-\omega t)}. \tag{13b}$$

This representation could now be substituted into Eq. (12a), and by imposing the boundary conditions (12b), re-expressed in terms of derivatives of the displacements, one would obtain the dispersion equations relating ω to k. However, for isotropic plates, it is conventional to first express the displacement $\mathbf{u}$ in terms of two scalar potentials, because the resulting wave equations for these potentials are de-coupled, whereas the equations for u_x and u_z are coupled. The displacement fields associated with these two potentials have a different character and different wavespeeds. One potential leads to equivoluminal motions ($\nabla.\mathbf{u} = 0$), with the displacement normal to the direction of propagation, i.e. to transverse waves, which have a wavespeed given by

$$c_T = (\mu/\rho)^{1/2}. \tag{14a}$$

The other leads to longitudinal waves ($\nabla \times \mathbf{u} = 0$), with

$$c_L = \{(\lambda + 2\mu)/\rho\}^{1/2}. \tag{14b}$$

The ratio of these two wavespeeds depends only on Poisson's ratio ν,

$$\kappa \equiv c_L/c_T = \{2(1-\nu)/(1-2\nu)\}^{1/2}. \tag{14c}$$

Thus, the displacement in Eqs. (13a, b) can be viewed as a superposition of two types of body wave, each with a well-defined wavespeed independent of the direction of propagation. These two types of wave are coupled through the boundary conditions (12b), leading to the following dispersion relation

$$\frac{\tan(\beta h)}{\tan(\alpha h)} = -\left[\frac{4k^2\alpha\beta}{(\beta^2 - k^2)^2}\right]^{\pm 1}, \tag{15a}$$

$$\alpha = \sqrt{\omega^2/{c_L}^2 - k^2}, \tag{15b}$$

$$\beta = \sqrt{\omega^2/{c_T}^2 - k^2}, \tag{15c}$$

where the choice of exponent in (15a) corresponds to symmetric and anti-symmetric modes respectively (Achenbach, 1973).

Although these equations were first derived by Rayleigh and Lamb a century ago, a comprehensive characterization of the resulting dispersion curves has not been achieved until comparatively recently, through the work of Mindlin (1960). The corresponding dispersion relation for waves in orthotropic plates (Habeger *et al.*, 1979, Eq. (20)) has the same structure as (15a), but its derivation is significantly more complicated, so that it will be worthwhile discussing the properties of (15a) first.

3.1 Summary of results for isotropic plates

The dispersion curves corresponding to (15a) can be constructed by considering a fixed value of the wavenumber k (>0) and solving (15a) for the corresponding values of angular frequency $\omega(k;n)$ where the integer n serves to index the possible solutions, which correspond to the various permissible modes of vibration. There are in fact two classes of solutions, associated with deformation states which are either symmetrical or anti-symmetrical with respect to the plate's mid-plane $z = 0$. For each of these two classes, there is an infinite number of possible modes. The two lowest-order modes, corresponding to $n = 0$, will be denoted by S_o and A_o, respectively.

In attempting to find the roots of (15a), it is useful first to identify three sectors in the first quadrant of the $\omega - k$ plane, as follows:

$$0 < \omega/k < c_T, \qquad \alpha,\ \beta \text{ imaginary;} \tag{16a}$$

$$c_T < \omega/k < c_L, \qquad \alpha \text{ imaginary, } \beta \text{ real;} \tag{16b}$$

$$c_L < \omega/k < \infty \qquad \alpha,\ \beta \text{ real.} \tag{16c}$$

The solutions for the two zero-order modes, S_o and A_o, fall within the first sector, described by (16a), so that the tan functions in (15a) are effectively tanh functions with a real argument. To proceed systematically, it would be appropriate at this stage to define non-dimensional variables. However, for the present purpose of summarizing the results, it seems preferable to retain the original physical variables.

The important features of the dispersion curves are (i) the behaviour for long and short wavelengths ($kh << 1$ and $kh >> 1$, respectively), and (ii) the cut-off frequencies for higher-order modes ($n \geq 1$).

Consider first the zero-order modes S_o and A_o. The long-wavelength behaviour obtained by retaining only the leading terms in (15a) for $kh << 1$ can be shown to agree closely with the predictions of simplified engineering theories of plate waves (Mindlin, 1951). At the other extreme, for $kh >> 1$, the ratio on the left-hand side of (15a) tends to unity, and (15a) reduces to the equation for Rayleigh surface waves on a half-space. Various approaches can be used to construct analytical approximations for the dispersion curves, so as to reproduce correctly the asymptotic behaviour for long and for short wavelengths.

The phase velocity c is defined by

$$c = \omega/k. \tag{17a}$$

By using this relation, a dispersion curve can be equivalently presented as a relation between the phase velocity c and the angular frequency ω, with the wavenumber k as a parameter. This alternative presentation is used by Habeger *et al.* (1979) for the orthotropic plate.

The group velocity c_g, defined by

$$c_g = d\omega/dk, \tag{17b}$$

can be interpreted physically as corresponding to the speed for transport of energy in a dispersive-wave motion, and it is therefore the more physically meaningful wave speed, compared with the phase velocity. From (17a, b) it follows that these two wavespeeds are related by

$$c_g = c + k\, dc/dk. \tag{17c}$$

It can be shown that the group velocity c_g tends to the phase velocity c in the limit $kh >> 1$, and it follows from (17c) that the curve of c versus k, or equivalently of c versus ω, tends to a horizontal asymptote which corresponds to the surface-wave speed in a half-space.

At the other extreme, the cut-off frequencies for higher-order modes ($n \geq 1$) can be obtained by taking the limit $k \to 0$ in (15a). Given that k is the wavenumber in the direction of propagation (the x-direction), a plate vibration in the limit $k \to 0$ can be interpreted as a one-dimensional standing wave in the thickness direction (zd).

This summary of key results and physical interpretations for waves in isotropic plates provides a useful reference point for discussing the more complicated orthotropic case.

3.2 Results for orthotropic plates

The existence of orthotropic plate-waves can be explored by assuming the form given in (13a, b) for the components u_x, u_z of the elastic displacement, as for the isotropic case. The greater mathematical complexity encountered for an orthotropic medium can be attributed physically to the fact that the phase velocity for plane waves now depends on the direction of propagation, and while the equations of motion (12a) again lead to two possible wavespeeds for bulk plane-waves, as for the isotropic case, the associated deformations are no longer purely transverse or purely longitudinal, except when the direction of propagation coincides with a principal axis. Thus, it is no longer possible to introduce displacement potentials to de-couple the equations for u_x and u_z. There are nevertheless many similarities with the isotropic case.

The dispersion relation is now given by the following equation (Habeger *et al.* 1979), instead of (15a),

$$\frac{\tan(\beta h)}{\tan(\alpha h)} = \left[\frac{H_m G_p}{H_p G_m}\right]^{\pm 1}, \tag{18a}$$

$$(\alpha/k)^2 = -B + \sqrt{(B^2 - 4D)/2}, \tag{18b}$$

$$(\beta/k)^2 = -B - \sqrt{(B^2 - 4D)/2}, \tag{18c}$$

where the plus and minus sign in the exponent of (18a) again correspond to symmetric and anti-symmetric modes respectively, as for (15a); B, D, $G_{p,m}$, $H_{p,m}$ are given by complicated expressions involving the orthotropic elastic constants which will not be reproduced here. It is sufficient to note that the right-hand side of (18a) depends only on the wavenumber k (or, more precisely, on the non-dimensional combination kh), whereas α, β depend in addition on the angular frequency ω, as in the case of (15b, c). Thus, the dispersion curves can again be constructed by solving (18a) for ω as a function k, and it is again convenient to divide the first quadrant into three sectors, depending on the sign of α^2 and β^2, as in (16a–c). The cut-off frequencies for higher-order modes can again be obtained by setting $k \to 0$, and the resulting motions interpreted as standing waves in the thickness direction (zd). This highlights the importance of the two bulk-wave speeds in the thickness direction, given by

$$c_T = (C_{55}/\rho)^{1/2}, \tag{19a}$$

$$c_L = (C_{33}/\rho)^{1/2}, \tag{19b}$$

where C_{33}, C_{55} are two of the four elastic constants required to characterize plane-strain deformation in an orthotropic material, with its principal axes aligned to the coordinate axes as assumed here, while ρ denotes the material's density. The wavespeeds in (19a, b) correspond respectively to transverse and longitudinal waves, as indicated by the subscripts. These equations would reduce to the earlier expressions (14a, b) for the special case of an isotropic material. It is noted that when the dispersion curves are plotted as phase velocity c versus frequency ω, as in Habeger *et al.* (1979), the cut-off frequencies can be readily identified by the fact that the curves exhibit a vertical asymptote as these frequencies are approached from above.

For the purposes of deriving material properties from on-line ultrasonic measurements, however, it would appear to be best to concentrate on the lowest-order modes, S_o and A_o, which have no cut-off frequency. In particular, the long-wavelength behaviour of the A_o mode should provide a directly useful measure of the bending stiffness, as noted in Section 1. A straight-forward expansion for $kh << 1$, as given by Habeger *et al.* (1979), leads to the standard relation for wavespeed versus frequency that would be obtained from engineering plate theories. It was pointed out at the MISG workshop that this long-wavelength (or low frequency) behaviour could therefore be used to define a *sonic flexural rigidity* as follows,

$$S_b = W\frac{c^4(\omega; A_o)}{\omega^2}, \tag{20}$$

where W denotes the basis weight and $c(\omega; A_o)$ the wavespeed for the A_o mode. This definition is analogous to that of the sonic extensional stiffness in (3), but it does not appear to have been used previously in the present context.

The exact representation which is available from Habeger *et al.* (1979) for the thickness variation of the stress components for given vibration modes could also be used to examine systematically the possible effects of pronounced anisotropy discussed in Section 2. In particular, this representation suggests that the right-hand side of (18c) could provide a more systematic definition for a non-dimensional index of anisotropy η, than the previous line of argument leading to (10). The significance of this parameter is that an entirely different combination of elastic constants may be involved if one considers the limiting form for large η, *before* deriving an asymptotic expansion for $kh << 1$. This point deserves further investigation.

Finally, it is noted that in the short-wavelength limit, the left-hand side of (18a) tends to unity, as in the case of (15a), and the resulting equation for the surface-wave speed along a principal direction (*md* in the present case) represents a generalization of the Rayleigh equation for surface waves on an isotropic half-space. It is possible to construct analytical approximations for the dispersion curves over the entire frequency range by devising suitable methods of reproducing the exact asymptotic behaviour for short and for long wavelengths. This could be achieved by adapting the approach used by Mindlin (1951) for isotropic plates, or, more simply, by postulating a suitable form for an interpolating function. The latter approach, using a rational function (Padé approximant), has been found to give results which are within 5% of the exact dispersion curve for a choice of elastic constants representative of chipboard (J. Ennis-King, private communication, March 1994). Such analytical approximations would greatly facilitate the task of reconstructing elastic stiffness coefficients from on-line measurements of wavespeed, while retaining an accurate account of the effects of pronounced anisotropy, and they deserve further investigation.

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This paper implicitly includes the contributions of the team of people who worked on the Amcor problem. The sections of the report largely correspond to the phases of the MISG workshop. In the initial phase, discussions between Russell Allan, the Amcor representative, and the rest of the team led to the conclusion that the determination of the bending stiffness S_b was the crucial structural parameter, rather than the effective thickness. In the second phase, analysis focused on the stiffness tests used in practice, particularly the Taber test and the Lhomargy test. Francis Rose led this part of the effort, with Russell Allan providing enthusiasm, data and practical details. A subgroup including Warren Wood and students also investigated the possible extensions to a theory of flexure for anisotropic plates. The third and coexisting phase centered on the analysis of the dispersion curves for plate waves, starting from the work of Habeger *at al.* (1979). Malcolm Anderson produced a simple analytical approximation for the location of the cutoff frequencies in terms of the elastic constants. It became clear from further discussion, including a passing contribution from Neville Fowkes, that the long wavelength behaviour of the lowest antisymmetric mode A_0 was probably the most sensible and physical way to obtain a measure of the bending stiffness. In work post-MISG Jonathan Ennis-King derived an analytic approximation for this mode. Contributions were also made by Joseph Ha, Peter McKinlay, Tim Passmore, Jacqueline Postle and Bernie Wendlandt.

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