Report 5

Modeling Compressible Non-Newtonian Chicken Flow

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Abstract

This paper addresses a few modeling issues relevant for the basic theoretical understanding of the meat flow behavior in simple geometries. We model the meat mixture as a non-Newtonian compressible fluid. Focusing on conceptually easy-to-follow cases like flow in thin molds, or steady incompressible or compressible flow in straight pipes we derive explicit expressions for the velocity and pressure profiles. For the thin mold case, we formulate a one-dimensional free-boundary problem able to capture the *a priori unknown* position of the moving meat-air interface. Special attention is payed on the derivation of the free boundary conditions.

5.1 Introduction

Understanding how meat flows is one of the fundamental aspects when designing a stable shape and content quality for food products, such as nuggets, croquettes, or meatballs. The overall process has a twofold complexity:

(1) Meat is a compressible non-Newtonian fluid with variable viscous properties and micro-structure (e.g. fiber orientation) strongly dependent on temperature variations. Such a flow behavior typically causes complex (meat) deformations especially in non-continuous flows, where the values of meat parameters and even the equipment itself never stabilizes. This is a highly complex scenario and complexity hampers the accurate prediction of both flow and final product quality (and, consequently, also the optimization of the processing equipment).

(2) The geometry (patterned manifolds, irregular molds) is often complex and is continuously changing from a product to another.

The problem posed by Marel to the *72 European Study Group Mathematics with Industry* was the following: Predict in a better way how meat properties affect flow in forming (molding) machines, where the meat mass is pressed in molds during mold opening and flow is a start-stop phenomena. More precisely, develop a mathematical model that predicts non-continuous flow of viscoelastic, compressible meat mass in simple geometries, where the pressure fluctuation, deformation rates, mold filling rates, and final product weight are key parameters.

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We have chosen to discuss the case of a simple geometry[4] – cylindrical pipes – and we have focused on developing a complete mathematical model for isothermal viscous compressible flow of meat (e.g. chicken). We included the micro-structural meat properties in a nonlinear power-law relationship connecting the shear rate to the shear stress. Besides the law governing the conservation of meat mass, we derived, by means of *first principles* arguments, a constitutive law for the meat density as a function of internal pressure.

In the steady case and also when neglecting the nonlinear inertia terms, we succeeded to find an approximate solution for the velocity profile for meat flow in cylindrical pipes and meat flow between two plates.

Furthermore, we can give theoretical estimates of the time to fill a mold in two conceptually-distinct ways:

(a) As mentioned in Remark 5.3.1, we know how to formulate a time-optimal-control problem for the meat density, the time to fill a mold, and a corrector factor (using an approximate velocity profile). Interestingly, the resulting problem resembles the porous media equation.

(b) We can suggest a calculation strategy, which gives exact results in one-space dimension, for the time to fill a mold, namely a free-boundary problem having as unknowns the velocity profile and the position of the interface between meat and air.

In our opinion, both working strategies deserve further attention from a combined modeling, analysis, and simulation perspective. Here we focus only on strategy (b).

The paper is organized as follows: In section 5.2, we develop a general model involving partial differential equations (PDEs) to describe the meat (chicken) flow. This is the core of our paper. The aim of section 5.3 is mainly to derive an easy-to-handle approximate solution for velocity profiles, for getting some insight about the characteristics of the problem, while in section 5.3 we propose a free-boundary problem to better understand meat flow behavior in linear molds.

5.2 Modeling chicken flow

In the current section we describe a complete set of equations which are able to capture the macroscopic behavior of chicken flow in a given geometry, say $\Omega \subset \mathbb{R}^3$. Note that the meat is a mixture of material fractions with different properties, such as a fibers, animals fats, bubbles of air trapped during the process of homogenization of the mixture, and so on. The presence of all these components, some of which having complex rheological properties, define the overall flow properties of the material. Assuming that the material is homogenized to an extent such that inhomogeneities of the meat are not noticeable at the macro-scale, allows us to make use exclusively of "effective" or averaged variables, coefficients and model equations. Let's denote by

[4]Note that the structure of the balance equations does not depend on the precise choice of the geometry.

T the final time of the process. Then $t \in (0,T)$ is the time variable, while $x \in \Omega$ is the spatial variable.

The conservation of mass is stated by continuity equation

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \mathbf{v}) = 0 \text{ in } \Omega \times (0,T). \tag{5.1}$$

Here ρ is the density of the meat-mixture, while $\mathbf{v}$ represents the meat flow velocity.

The balance of the (linear-) momentum density is described by a Navier-Stokes-like equation. The major difference here compared to the Navier-Stokes equations for usual Newtonian liquid lies in the very special expression of the stress tensor. This balance of linear momentum reads:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + (\mathbf{v} \cdot \nabla (\rho \mathbf{v})) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} \text{ in } \Omega \times (0,T). \tag{5.2}$$

Here $\boldsymbol{\sigma}$ is the stress tensor and p is the static pressure in the material.

Due to the specifics of our problem, all investigations reported in this note are focused on shear flows. Therefore it is enough to determine averaged shear properties of the meat-mass to account for the contributions to the stress tensor. Previous works (cf. e.g. [6, 8, 3, 4], or [7] (chapter 5)) indicate that the flow properties of the meat-masses can be taken in the form:

$$\eta_{shear} = k |\dot{\gamma}|^{n-1} \text{ in } \Omega \times (0,T), \tag{5.3}$$

where $k > 0$ and $n \in [0.2, 0.4]$ are empirical[5] coefficients.

To complete our model we still need an equation of state. To be more precise, we have to specify the compressibility properties of the material as a function of its actual density and temperature. The change of temperature has a twofold effect - it changes coefficients k, n and directly affects the flow compressibility. It is worth noting that the empirical equation (5.3) defining the shear viscosity of the meat flow is approximate, and moreover, this approximation strongly depends on temperature variations. This means that we expect that small changes in temperature are able to produce rather large deviations from the real shear viscosity. The parameters k and n play the role of correctors compensating some of the errors induced by the variations in temperature. In what we are concerned, we consider only an isothermal situation, hence, k and n are fixed for when fixing the temperature level. What about the influence of the temperature changes on the compressibility? The main part of the compressibility of the meat-mass is due to the most *compressible* material fraction, i.e. due to air bubbles. The effect is rather obvious: The bigger the bubble fraction is, the bigger the compressibility. Roughly speaking, in order to notice the compressibility of the air the changes of temperature should be in the range of $300K$. However, this temperature range is not the one encountered when filling molds with flow meat. This fact suggests that as equation of state we may consider a *density-pressure relationship*.

The fact that the our material system consists of both compressible and incompressible parts (air-filled parts *versus* liquid and solid parts of meat) leads us to the

[5] In most of the cases, n and k are fitting parameters. We expect them to incorporate important micro-structure information like the local orientation of the meat fibers.

following *first principles* approach:

$$p\mathrm{V} = \tilde{b}_1 p + \tilde{b}_2 \text{ in } \Omega \times (0,T), \tag{5.4}$$

or equivalently,

$$\rho = \frac{p}{b_1 p + b_2} \text{ in } \Omega \times (0,T). \tag{5.5}$$

Here b_1, b_2, or equivalently, $\tilde{b}_1, \tilde{b}_2$ are material coefficients depending on the concentration of air.

The equation of state (5.4) together with boundary conditions (describing the experimental setup – the concrete food-processing machine) and initial condition (the precise type of meat) complete the set of our model equations. It is worth noting that, trusting arguments like those employed, for instance, in [1] (chapter 3) and [2] (chapter 3), we expect that our model is *thermodynamically consistent* in the sense that it fulfills the Clausius-Duhem inequality.

In the remainder of the paper, we study various flow behaviors corresponding to specific (simple) geometries when boundary conditions delimitate different experimental situations. Our focus will then be oriented towards the motion of the *a priori unknown* free interface separating meat and bulk air.

5.3 Construction of approximate velocity profiles

In this section we derive an approximate velocity profiles for the chicken flow in cylindrical pipes. We start by considering the case of steady incompressible meat flow. Under these assumptions, the term $\frac{\partial}{\partial t}(\rho \mathbf{v})$ drops out, and hence (5.2) takes the form

$$\rho(\mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} \text{ in } \Omega. \tag{5.6}$$

Due to the prominent viscous properties of the meat mixture and due to the particular scale of speed used in the processing machines, inertial effects become less important than viscous effects. Therefore, neglecting the inertia term in (5.2) yields

$$-\nabla p + \nabla \cdot \boldsymbol{\sigma} = 0 \text{ in } \Omega. \tag{5.7}$$

Note that (5.7) is some sort of Stokes-like approximation, in which the stress tensor $\boldsymbol{\sigma}$ appears in general form.

In what follows we consider particular geometries mimicking standard ones used in food processing technologies in order to derive closed-form expressions for meat velocity and density profiles.

Steady flow in straight pipe

Let us consider firstly a straight cylindrical pipe. In this particular case, by neglecting the effects of gravity, we can assume axial symmetry of the flow with respect to the center line of the pipe. Consequently (5.7) can be rewritten in cylindrical coordinates as follows

$$-\frac{\partial p}{\partial z} + \frac{\partial}{\partial r}\left(k|\dot{\gamma}(r)|^{n-1}\dot{\gamma}(r)\right) + \frac{1}{r}k|\dot{\gamma}|^{n-1}\dot{\gamma}(r) = 0, \text{ in } \Omega, \tag{5.8}$$

where z is the coordinate in the axial direction, and r is the coordinate in the radial direction. Applying the chain rule of differentiation and multiplying the result by r gives

$$-r\frac{\partial p}{\partial z}+\frac{\partial}{\partial r}\left(rk|\dot{\gamma}(r)|^{n-1}\dot{\gamma}(r)\right)=0 \text{ in } \Omega. \tag{5.9}$$

After integrating with respect to r and substituting $\dot{\gamma}(r)=-\partial v/\partial r$, where v denotes the axial velocity component, the latter equation takes the form

$$-\frac{r}{2k}\frac{\partial p}{\partial z}+\left|\frac{\partial v}{\partial r}\right|^{n-1}\frac{\partial v}{\partial r}=0 \text{ in } \Omega. \tag{5.10}$$

Solving for $\partial v/\partial r$, integrating with respect to r, and finally, using the no-slip condition at the wall of the pipe, we obtain

$$v(r)=\frac{n}{n+1}\left(-\frac{1}{2k}\frac{\partial p}{\partial z}\right)^{1/n}\left(R^{\frac{n+1}{n}}-r^{\frac{n+1}{n}}\right) \text{ for all } r\in\Omega, \tag{5.11}$$

where the new constant R stands for the radius of the pipe. Fig. 5.3 shows a comparison between the well known parabolic Poiseuille profile for Newtonian fluids and a typical profile for steady non-Newtonian flow (5.11). Depending on the particular value of the parameter n, the profile becomes flatter or steeper. For the case of chicken flow with $n\in[0.2,0.4]$ the viscosity decreases when the shear rate increases. Consequently, the profile becomes flatter.

Compressible steady flow in a pipe

The velocity profile (5.11) was obtained by neglecting the effects of compressibility. However, when speaking about meat flow, one actually wants to keep some compressibility effects in the game. In this section, we look at a steady flow of the meat-mixture and including the compressibility accordingly to the equation of state (5.4).

In order to simplify matter, we consider our equation already averaged over the cross section of the pipe. Let $\hat{\Omega}:=(0,L)$ be the new domain, where L is the length of the pipe. For notational convenience, we use in the derivations below v for denoting the average velocity over the pipe cross section. Note that the velocity v can be decomposed as

$$v(z)=v_0+u(z), \tag{5.12}$$

where v_0 is the average of the velocity profile (5.11) derived in the previous subsection. Furthermore, by neglecting inertia effects and linearizing our equations, we obtain the following system of equations posed in $\hat{\Omega}$:

$$\frac{\partial p}{\partial z}+\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial z}=0, \tag{5.13a}$$

$$\rho\frac{\partial u}{\partial z}+v_0\frac{\partial \rho}{\partial z}=0. \tag{5.13b}$$

Dividing (5.13b) by ρ and integrating with respect to z we obtain

$$u(z)=-v_0\ln\left(\frac{\rho}{\rho_0}\right). \tag{5.14}$$

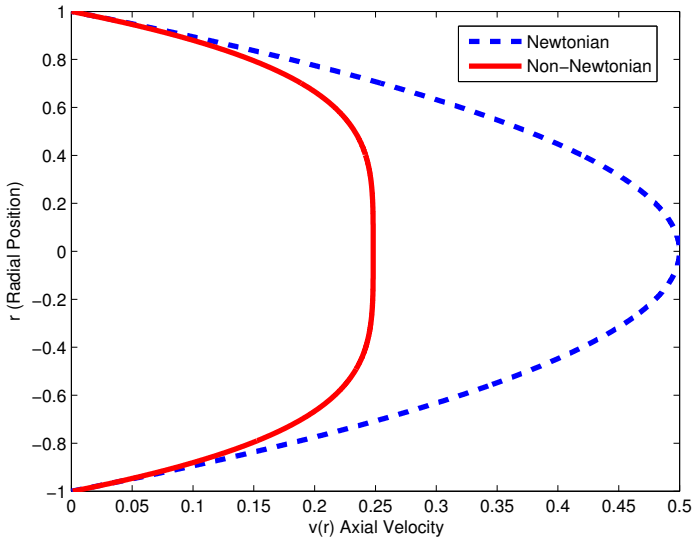

Figure 5.1: Velocity profiles in a straight pipe, for Newtonian and non-Newtonian fluid.

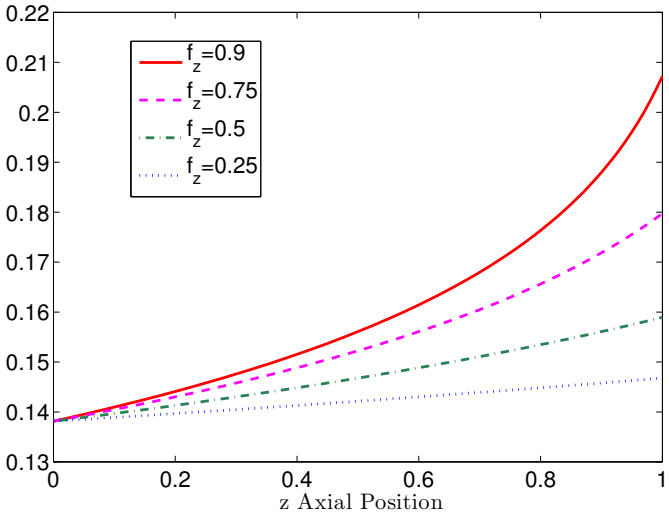

Figure 5.2: Velocity deviation u as function of the axial position.

From (5.13a) we obtain then the density $\rho(z)$. Indeed, substituting (5.4) in (5.13a) yields

$$\frac{\partial}{\partial \rho}\left[(b_1 p + b_2)\rho\right] = \frac{\partial p}{\partial s}, \tag{5.15}$$

from which we finally obtain

$$\rho(z) = \frac{1 - \frac{\partial p}{\partial z} z}{b_1 \left(1 - \frac{\partial p}{\partial z}\right) z + b_2} \text{ for all } z \in \hat{\Omega}. \tag{5.16}$$

One important conclusion can be observed from this derivation. That is, that even in the case of steady flow, the average velocity along the pipe is not uniform. In fact, due to compressibility, the velocity of the flow increases with the axial direction. This can be clearly observed in Fig. 5.2 for different values of the pressure gradient, the density is plotted in Fig. 5.3. The decompression taking place in the axial direction, introduces an extra velocity component to the flow, making the velocity distribution non-uniform.

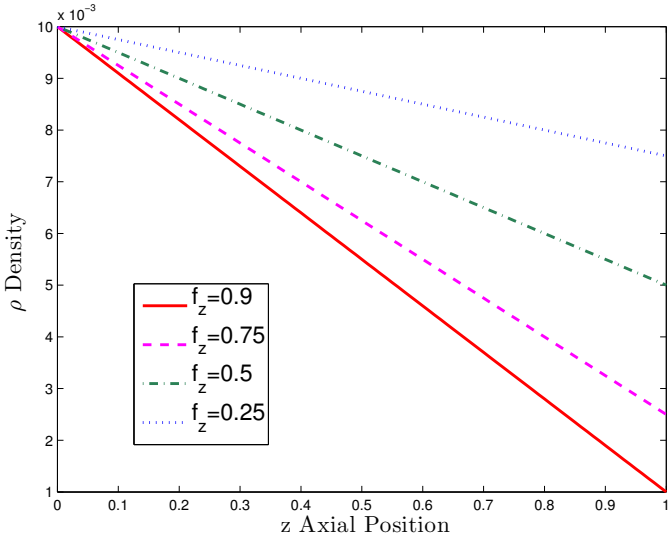

Figure 5.3: Density ρ as function of the axial position $z \in \hat{\Omega} := (0, L)$ with $L = 1$.

Remark 5.3.1. Interestingly, the pipe chicken flow situations described is remotely resembling the porous media equation. One can see this easily when inserting the explicit expressions of $v = v(\rho, \rho_0, v_0)$ into meat mass balance. This similarity can potentially be used (see e.g. [5]) to formulate a time optimal control problem for chicken flow.

On moving free chicken-air interfaces in linear molds

The aim of this part is to describe the filling of the mold by meat masses. We assume that the mold has the form of the parallelepiped with a much less height compared to its width and thickness. We choose the Cartesian coordinate system so that Oz is directed vertically, Ox is directed from left to the right and Oy forms the right triple with the other two axes.

Near the walls the speed of the meat is minimal and somewhere inside gains its maximum. The small height of the mold means that the vertical velocity gradients are much bigger than the horizontal ones. Consequently the main input in the friction term seems to be caused by vertical velocity gradients. If the boundary data prescribed at the left and right sides do not vary much along the sides then it also makes sense to formulate the filling-mold-problem as a one-dimensional event. Let's denote by $s(t)$ the free boundary (meat front) separating the meat bulk $(0,s(t))$ from the air part $(s(t),L)$. The front of the meat starts to propagate from left side to the right. We are now interested in the time-behavior of the front of the meat. The equations in this case can be derived from equation (5.2) by integrating along the Oy and Oz directions. Locally, due to the high viscosity of the meat mixture, the velocity profile along Oz direction can be taken as in-between two parallel plates. But, of course, the average velocity given by this profile can be different for different points x.

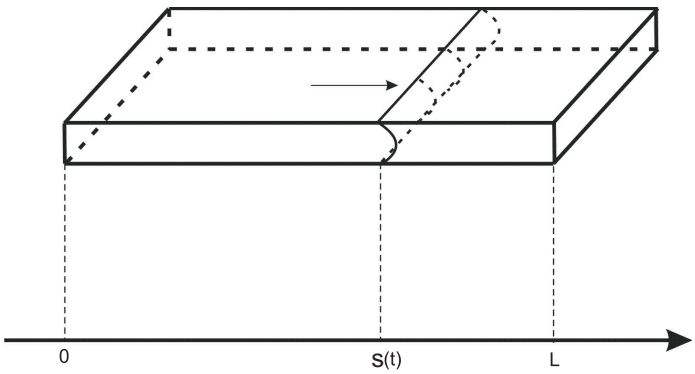

Figure 5.4: The velocity profile for the steady flow between two plates.

The velocity profile between two parallel plates is then

$$v(x)=\frac{n}{n+1}\left|\frac{\nabla p}{k}\right|^{\frac{1}{n}}\left(\left(\frac{d}{2}\right)^{\frac{n+1}{n}}-x^{\frac{n+1}{n}}\right)\ \text{for}\ x\in\left(-\frac{d}{2},\frac{d}{2}\right). \tag{5.17}$$

Here d is the distance between the plates, ∇p is the pressure gradient, while k and n enter the expression for the viscosity (5.3). From this profile we get a relation between the average velocity and the pressure drop due to viscosity

$$\nabla p_{_{drop}}=\xi|v|^{n-1}v, \tag{5.18}$$

where ξ is a constant coefficient which relates to the coefficient k.

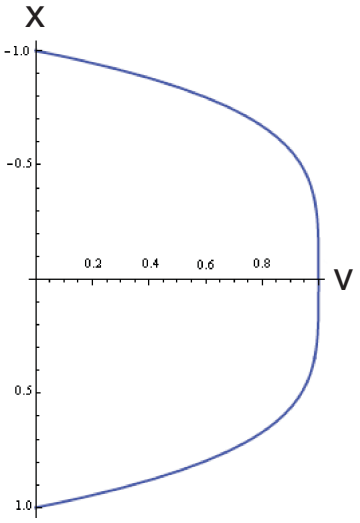

Figure 5.5: The velocity profile for steady meat flow between two plates.

Summarizing, we get the equations for the averaged velocity $v(x)$ and density $\rho(x)$ in the mold. The balance of momentum is

$$\frac{d(\rho v)}{dt} = -\frac{\partial p}{\partial x} - \xi |v|^{n-1} v \text{ in } [0,L]\times(0,T), \tag{5.19}$$

and the conservation of meat mass reads

$$\frac{d\rho}{dt} + \rho(\nabla \cdot v) = Q(x,t) \text{ in } [0,L]\times(0,T). \tag{5.20}$$

Here the total differentials are the Lagrange derivatives, i.e. this is the derivative in accompanying system of reference for a local portion of the fluid. The function $Q(x,t)$ - accounts for a possible sources of meat inside the mold. The additional sources of the meat are the possible inlets for the meat at the sides of the mold.

The boundary conditions are:

$$p(0,t) = p_{in}(t) \qquad \text{and} \qquad p(s(t),t) = p_b.$$

Here $v_{in}(t)$ is the velocity at the left side of the mold, $p_{in}(t)$ is the pressure on the left side, and $p(s(t),t) = p_b$ - the pressure at the meat front. p_b is the atmospheric pressure. In practice there are many small outlets for the air to go out in the mold. The inertia of the air is negligible compared to the inertia of the meat. That is why we can neglect the changes of the pressure near the right boundary due to the flow of air. $s(t)$ is the function of the coordinate of the meat-air boundary on time and is an unknown of our model. To determine $s(t)$ we have to introduce an additional

interface condition. This condition naturally arises when imposing the momentum balance at that boundary. But, as it was said above, we neglect the inertia of the air. Consequently, the momentum transfer by air is also negligible. So the condition on the free boundary should be "no momentum transfer to the air", i.e.

$$\rho\frac{d^2s}{dt^2}=-\frac{\partial p}{\partial x}-\xi\left|\frac{ds}{dt}\right|^{n-1}\frac{ds}{dt}\ \text{in}\ [0,L]\times(0,T). \tag{5.21}$$

This equation defines the motion of the free boundary. It is a kind of Rankine-Hugoniot conditions. But to make use of it we need to know the pressure gradient near the boundary. To find it we have to find the function $p(x,t)$. Therefore this equation has to be solved together with (5.19) and (5.20). If we assume for a moment, that the pressure gradient is constant in time, then we can immediately find the meat-boundary motion. The pressure on the moving boundary is constant, and therefore the density near the boundary is also constant. This leads to the following equation for the boundary

$$\rho\frac{dv}{dt}=\xi|u|^{n-1}u-\xi|v|^{n-1}v\ \text{in}\ [0,L]\times(0,T). \tag{5.22}$$

Here u is a constant that can be related to the strength of the flow. The bigger the pressure applied on the left edge, the bigger u is. The plot $v(t)=s'(t)$ is shown in Fig. 5.6.

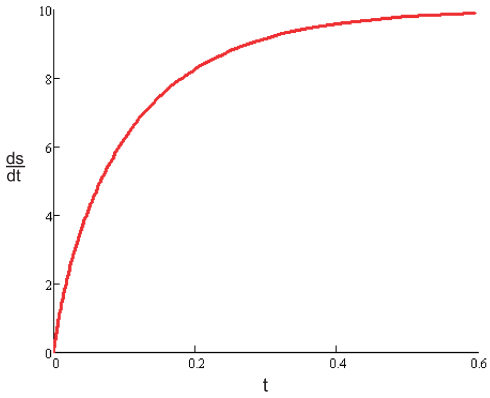

Figure 5.6: The velocity $s'(t)$ of the meat front in the mold as a function of time.

It seems that for short time the velocity of the meat front follows the asymptotic relation $s'(t)=\mathcal{O}\left(t^{\frac{1}{2}}\right)$ for $t>0$. On the other hand, we expect that the large time behavior of $s'(t)$ will essentially depend on the exponent n. We will focus elsewhere on capturing the precise short- and large-time asymptotics of the meat front.

Acknowledgments

Fruitful discussions with Han Slot (TU Eindhoven) are kindly acknowledged. We also thank Bertus Dunnewind (STORK) for introducing us to this subject.

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