Delaware MPI 2012 Report: Problem from Standard and Poor's

Ghan S. Bhatt (Tennessee State U.) John Brimlow (RPI)
Brian Estervig (Iowa State U.) Elena Caraba (U. Illinois)
Abouzar Ghavami (RPI) Shimao Fan (Temple U.)
Nguyenho Ho (WPI) Haijun Gong (CMU)
Ian Lenhoff (Cornell U.) Zhan Huang (Penn State U.)
Chao Liang (Penn State U.) Jeffrey Sanders (T. Jefferson U.)
Shi Jin (UD) Ying Wang (U. Minnesota) Lou Rossi (UD)
Guangming Yao (Clarkson U.) Melody Takeuchi (Tufts U.)
Lei Yao (RPI) Gnana Tenali (FIT) Dean Duffy (NASA)
Katherine Varga (NC State U.) Pak-Wing Fok (UD)
Tong Zhao (Iowa State U.) Colin Please (U. Southampton)

June 11, 2013

Abstract

Modeling how the returns of assets in a portfolio are correlated is crucial to properly understand and price the risk in the portfolio. In this report we discuss methods for approximating a full correlation matrix with a k-factor model. We then discuss methods for efficiently simulating asset returns given such a model.

1 Introduction

A common approach to portfolio return modeling is to combine a process for change in individual exposure returns with a dependence structure across exposures that captures the joint evolution [1]. This dependence structure can be described by a correlation matrix, which will be a symmetric positive definite matrix with ones along the diagonal. The large number of assets in many portfolios of interest makes the size of the corresponding correlation matrix large which in turn makes simulation difficult. Hence approximating the given correlation matrix with a much lower dimensional factor model is suggested. This is equivalent to approximating a correlation matrix with a lower rank matrix. In this case, each asset return in the portfolio depends on a relatively small number of factors which remains fixed as the size of the portfolio increases. There are several factor models provided in [1].

To introduce factor models, suppose a portfolio has N assets and let us consider a factor model with N_F factors. The asset returns $\mathbf{z} = (z_1, z_2, \cdots z_N)^T$ is what we wish to model. Let ρ_i be the percentage of asset return variance associated with systematic risk and let the column vector $\boldsymbol{\beta}_i = (\beta_{i,1}, \beta_{i,1} \cdots \beta_{i,N_F})^T$, be factor loading vector that is normalized so that $\boldsymbol{\beta}_i^T \boldsymbol{\beta}_i = 1$. Let $\boldsymbol{\epsilon}_F = (\epsilon_{F,1}, \epsilon_{F,2}, \cdots \epsilon_{F,N_F})^T$ represent the returns for the assets within each factor. The $\epsilon_{F,i}$ are assumed to be identically distributed standard normal variables represented by the orthogonalized factor returns. Let $\boldsymbol{\epsilon}_I = (\epsilon_{I,1}, \epsilon_{I,2}, \cdots \epsilon_{I,N})^T$ represent the idiosyncratic risk not explained by the factors and assumed independent of the factor returns; generally they are assumed independent across all assets in the portfolio. The asset return is given by

$$\mathbf{z} = \Gamma^{1/2} B \boldsymbol{\epsilon}_F + (I - \Gamma)^{1/2} \boldsymbol{\epsilon}_I,$$

where B is a $N \times N_F$ matrix called the loading matrix,

$$B = \begin{pmatrix} \boldsymbol{\beta}_1^T \\ \boldsymbol{\beta}_2^T \\ \vdots \\ \boldsymbol{\beta}_N^T \end{pmatrix}$$

and Γ is a $N \times N$ diagonal matrix with ρ_i on the diagonal. Since $\boldsymbol{\epsilon}_F, \boldsymbol{\epsilon}_I \sim N(0, 1)$ the associated correlation matrix $E(zz^T)$ can then be written as

$$P = \Gamma^{1/2} B B^T \Gamma^{1/2} + I - \Gamma.$$

This correlation matrix has N_F factor correlation matrix structure. Authors in [1, 2] have shown that there exists optimal Γ and hence optimal B in the sense of Frobenius norm. The first part of this report will discuss the problem of determining the factor loadings β_i. The second part of this report will discuss ways of efficiently simulating the returns z given the factor loadings.

2 Optimal k-Factor Approximations

One method of approximating a given correlation matrix with a k-factor model is to use the largest k principal components. In this section we will discuss methods for obtaining this approximation and the error involved.

2.1 Approximations Using Full Matrix

One method for determining the factor loadings of a k-factor model based on principal components is iterative using the known full correlation matrix P. It is based on the equation

$$P = \Gamma^{1/2} B B^T \Gamma^{1/2} + I - \Gamma.$$

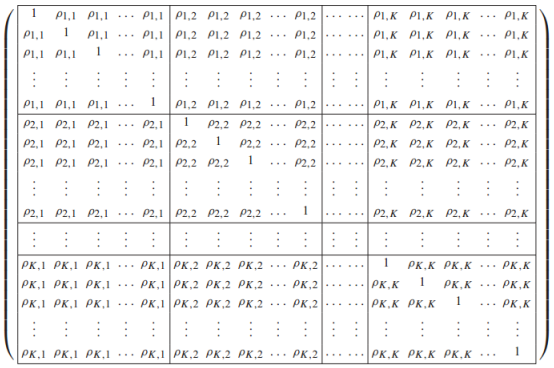
We start with an initial guess Γ_0, then at each step perform an eigenvector decomposition on $P - (I - \Gamma_i) = Q_i D_i Q_i^T$ where Q_i is an orthogonal matrix and D_i is diagonal . We then

"clip" D_i by replacing the smallest $N-k$ diagonal entries of D_i by 0, resulting in a clipped matrix $D_i^{(k)}$. The gamma for the next iteration will then be $\Gamma_{i+1} = \text{diag}(Q_i D_i^{(k)} Q_i^T)$. This is known to converge to an optimal $\Gamma_{\text{opt}}^{(k)}$ [1]. The optimal factor loadings can then be found by constructing

$$B_{\text{opt}}^{(k)} = \left(\Gamma_{\text{opt}}^{(k)}\right)^{-\frac{1}{2}} Q_{\text{opt}} \left(D_{\text{opt}}^{(k)}\right)^{\frac{1}{2}}.$$

2.2 Approximations Using Block Correlation Matrices

A direct method to solve for the factor loadings can be devised by assuming a block structure of the correlation matrix. In a block correlation matrix with n groups, assets in the same group have the same correlations.



The reduced form is an $n \times n$ matrix with one entry for each group of the block matrix.

$$\begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \cdots & \rho_{1,N} \\ \rho_{2,1} & \rho_{2,2} & \cdots & \rho_{2,N} \\ \vdots & \cdots & \ddots & \vdots \\ \rho_{N,1} & \rho_{N,2} & \cdots & \rho_{N,N} \end{pmatrix}$$

The reduced correlation matrix with ones on the diagonal is P_g, and Γ_g has elements $\rho_i + (1-\rho_i)/N_i$, where N_i is the number of assets in group i. We can write

$$P_g = \Gamma_g^{1/2} B B^T \Gamma_g^{1/2} + I - \Gamma_g, \tag{1}$$

where B is an $n \times n$ matrix whose rows are the factor loadings.

Using this reduced matrix we can formulate a direct method for calculating the factor loadings.

We start by solving the following equation for BB^T :

$$P_B = BB^T = \Gamma_g^{-1/2}(P_g - I + \Gamma_g)\Gamma_g^{-1/2}. \tag{2}$$

Using an eigenvector decomposition we get $P_B = QDQ^T$. A k-factor approximation is obtained by keeping the largest k eigenvalues of D, and setting the rest to zero. As above,

new "clipped" diagonal matrix is denoted by $D^{(k)}$, and the k-factor approximation of B is given by

$$B^{(k)} = V\sqrt{D^{(k)}} \tag{3}$$

We can write the k-factor approximation of P_g as $P_g^{(k)} = B^{(k)}(B^{(k)})^T$.

2.3 Error Analysis for the k-Factor Approximations

To compare the k-factor approximation of our correlation matrix to the original correlation matrix, we found an upper bound for the relative error:

$$\text{Relative Error} = \frac{||P - P^{(k)}||}{||P||} \leq \frac{\sqrt{\lambda_{k+1}}}{\sqrt{\lambda_{max}(P^T P)}}, \tag{4}$$

where λ_{k+1} is the largest eigenvalue of $(D - D^{(k)})^T(D - D^{(k)})$. Similarly, for the block correlation matrix P_g:

$$\text{Relative Error} = \frac{||P_g - P_g^{(k)}||}{||P_g||} \leq \frac{||\Gamma_g^{1/2}||^2\sqrt{\lambda_{k+1}^g}}{\sqrt{\lambda_{max}(P_g^T P_g)}}, \tag{5}$$

where λ_{k+1}^g is the largest eigenvalue of $(D-D^{(k)})^T(D-D^{(k)})$. This means that the actual error is bounded above by the largest eigenvalue that was set to zero in the k-factor approximation.

The error depends on the spread of eigenvalues. To see this, let P_g be a 20×20 matrix with eigenvalues distributed as shown below

The largest eigenvalue is about 2.8, the next two are close to 0.4, and the remaining eigenvalues are clustered around 0.07. The following table shows the relative error we computed for the specified approximations.

kth order approx.	Relative error
19	.15
4	.16
3	.16
2	.22
1	.23

Notice from the table that the error does not change much as the smaller eigenvalues are "clipped," or set to zero. However, once the approximations start to clip off the larger eigenvalues, we see a jump in the error.

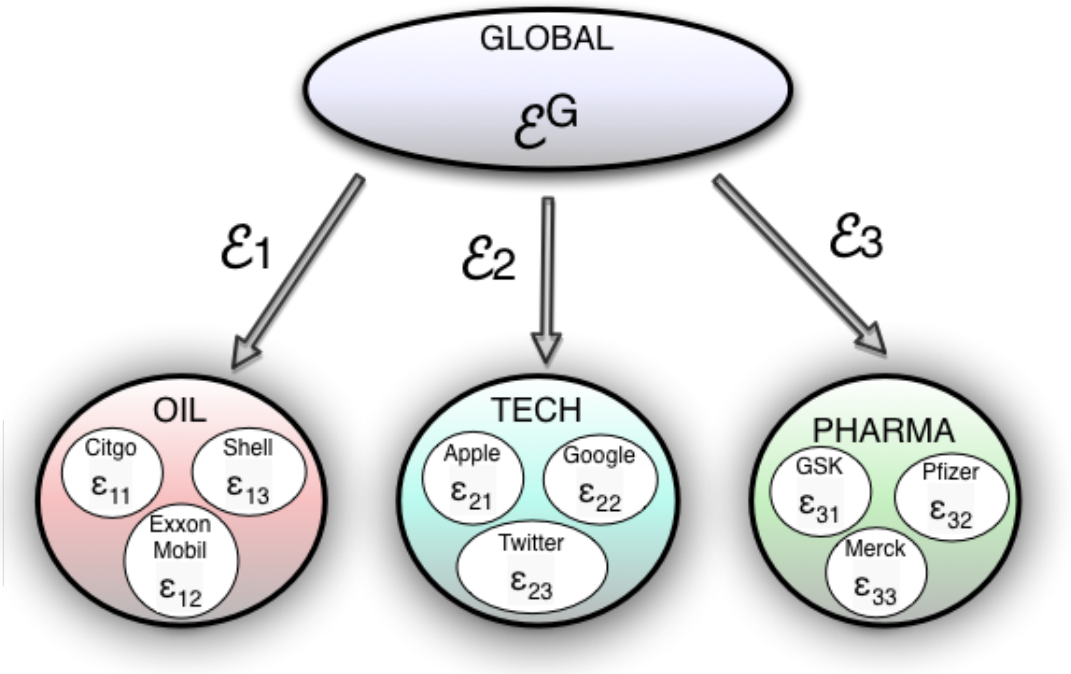


Figure 1: Asset returns of a companies in a one-factor model are affected by sector and global factors.

3 Localized One Factor Model

If the assets can be divided into n sectors (as in the terminology of [1]), and each asset weigh on one common factor shared by all assets (called the global factor), and only on one other factor drawn from a group of n factors, then the above factor model species that the asset return is influenced by a global factor, a sector-specific factor and its idiosyncratic risk. The resulting factor model is called a localized one factor (because one factor is chosen from each sector) model.

3.1 Portfolio Pricing in a Localized 1-Factor Model

Assume that there are n_i companies in sector i. Assume there are N sectors altogether. The asset return of the jth company in sector i is given by

$$z_{ij} = \sqrt{\rho_i}(\beta_G \epsilon_G + \beta_i \epsilon_i) + \sqrt{1-\rho_i}\phi_{ij}, \qquad i = 1, \ldots, N, \quad j = 1, \ldots, n_i. \tag{6}$$

where $\beta_G^2 + \beta_i^2 = 1$. In eq. (6), ρ_i is the percentage of asset return variance associated with systemic risk, β_G and β_i are global and sector factor loadings and ϵ_G, ϵ_i and ϕ_{ij} are standard normal random variables. ϵ_G and ϵ_i are represent uncertainty within the global economy and individual sectors while ϕ_{ij} is the *idiosyncratic factor* that characterizes an individual company's performance.

The value of the portfolio is

$$R_\pi = \sum_{i=1}^{N} \sum_{j=1}^{n} \hat{w}_{ij} R_i(z_{ij}), \tag{7}$$

where $\sum_{i=1}^{N} \sum_{j=1}^{n} \hat{w}_{ij} = 1$. The company defaults if the stock price z_{ij} drops below a value θ_i which depends on the sector. At default, the loss incurred in the portfolio is LGD (Loss Given Default):

$$R_i(z_{ij}) = \begin{cases} 0 & \text{if } z_{ij} > \theta_i, \\ -LGD & \text{if } z_{ij} < \theta_i. \end{cases} \tag{8}$$

In this report, the LGD can take one of two forms. Either $LGD = c$ corresponding to a constant percentage loss ($0 < c \leq 1$) or LGD can be drawn from a uniform distribution $LGD \sim U(0, 1)$.

Our goal is to find the probability distribution of the random variable R_π. Specifically, given a (small) probability α, we wish to find the threshold price R^* for which

$$P(R_\pi < R^*) = \alpha, \tag{9}$$

which amounts to computing the tail probability of the pdf (probability density function) of R_π, which we call $f_{R_\pi}(R^*)$.

3.2 Conditioning to obtain independence

The losses for each company $R_i(z_{ij})$ in eq. (7) are not statistically independent since all companies in sector i are affected by a single sector factor ϵ_i. Therefore the losses for these companies are correlated at the sector level. Furthermore, all companies are affected by a single global factor ϵ_G; therefore the losses for all companies in the global economy are correlated.

Our strategy is to condition on the sector and global factors, in effect holding ϵ_i and ϵ_G constant for all i. Then, the uncertainty in the stock price of all companies only arises from the idiosyncratic portion of eq. (6) and are therefore independent. This allows the use of the Central Limit Theorem in order to find the conditional pdf of R_π, $f_{R_\pi|\vec{\epsilon},\epsilon_G}(R^*; \vec{\epsilon}, \epsilon_G)$. The full pdf of R_π is then found by performing a $N + 1$ dimensional integral over each of the sector factors and the global factor:

$$f_{R_\pi}(R^*) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_{R_\pi|\vec{\epsilon},\epsilon_G}(R^*; \vec{\epsilon}, \epsilon_G) g(\vec{\epsilon}) \mathrm{d}\vec{\epsilon} \mathrm{d}\epsilon_G. \tag{10}$$

where

$$\vec{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_N)^T, \tag{11}$$

$$\mathrm{d}\vec{\epsilon} = \mathrm{d}\epsilon_1 \mathrm{d}\epsilon_2 \dots \mathrm{d}\epsilon_N, \tag{12}$$

$$g(\vec{\epsilon}) = \frac{1}{(2\pi)^{N/2}} \prod_{i=1}^{N} e^{-\epsilon_i^2/2} \tag{13}$$

3.3 Probability of Default

For every sector i, there is a default threshold θ_i so that if the asset return for a company in that sector drops below θ_i, the company defaults. The probability of default for a company in sector i is

$$p_i(\epsilon_i, \epsilon_G; \theta_i, \beta_i, \rho_i) = P(z_{ij} < \theta_i) = \Phi\left[\frac{\theta_i - \sqrt{\rho_i}(\beta_G \epsilon_G + \beta_i \epsilon_i)}{\sqrt{1-\rho_i}}\right]. \tag{14}$$

where $\Phi(z) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$.

3.4 Application of Laplace's Method

We make the following assumptions to simplify the calculation:

- The number of companies in each sector is the same, so $n_i = n$ for all i.
- The weight put on all companies in a given sector is the same so that $\hat{w}_{ij} = \hat{w}_i$ for all $j = 1, \ldots, n$.
- The number of companies in each sector is large: $n \gg 1$.
- Loss Given Default is a constant: $LGD = c$.

Under these assumptions, the value of our portfolio is

$$R_\pi = \sum_{i=1}^{N} X_i, \qquad X_i = \frac{w_i}{N}\left(\frac{R_i(z_{i1}) + \ldots + R_i(z_{in})}{n}\right). \tag{15}$$

where we have renormalized the weights so that $\hat{w}_i = \frac{w_i}{nN}$ and $\sum_{i=1}^{N} w_i = N$. Note that $w_i = O(1)$ for all i.

For constant $LGD = c$, it is clear that $R_i(z_{ij}) \sim -c \times \text{Bernoulli}(p_i)$ are proportional to Bernoulli random variables (RVs) and (15) implies that X_i is proportional to the mean of n such RVs. Since $n \gg 1$, we can use the Central Limit Theorem to deduce that

$$\frac{w_i}{N}\left(\frac{R_i(z_{i1}) + \ldots + R_i(z_{in})}{n}\right) \sim \mathcal{N}\left(\frac{-cw_i p_i}{N}, \frac{c^2 w_i^2 p_i(1-p_i)}{nN^2}\right). \tag{16}$$

Therefore

$$R_{\pi|\vec{\epsilon},\epsilon_G} \sim \mathcal{N}(\hat{\mu}, \hat{\sigma}^2) \Rightarrow f_{\pi|\vec{\epsilon},\epsilon_G}(R^*) = \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \exp\left[-\frac{(R^* - \hat{\mu})^2}{2\hat{\sigma}^2}\right], \tag{17}$$

$$\hat{\mu}[\vec{\epsilon}, \epsilon_G] = -\frac{c}{N}\sum_{j=1}^{N} w_j p_j[\epsilon_j, \epsilon_G], \tag{18}$$

$$\hat{\sigma}^2[\vec{\epsilon}, \epsilon_G] = \frac{\delta^2 c^2}{N^2}\sum_{j=1}^{N} w_j^2 p_j[\epsilon_j, \epsilon_G]\left(1 - p_j[\epsilon_j, \epsilon_G]\right), \qquad \delta = 1/\sqrt{n} \tag{19}$$

$$\equiv \delta^2 \tilde{\sigma}^2[\vec{\epsilon}, \epsilon_G]. \tag{20}$$

The number of sectors N does not have to be large. From (10), we have

$$f_{R_\pi}(R^*) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \exp\left[-\frac{(R^*-\hat{\mu})^2}{2\hat{\sigma}^2}\right] \times \exp\left[-\frac{1}{2}\left(\epsilon_G^2 + \sum_{j=1}^{N} \epsilon_j^2\right)\right] \frac{\mathrm{d}\vec{\epsilon}\,\mathrm{d}\epsilon_G}{(2\pi)^{\frac{N+1}{2}}}, \quad (21)$$

$$= \frac{1}{(2\pi)^{\frac{N}{2}+1}} \int_{-\infty}^{\infty} \mathrm{d}\epsilon_G \exp\left(-\frac{\epsilon_G^2}{2}\right) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[-\frac{\chi(R^*)}{2\delta^2}\right] \exp\left[-\frac{\vec{\epsilon}\cdot\vec{\epsilon}}{2}\right] \frac{\mathrm{d}\vec{\epsilon}}{\hat{\sigma}(\vec{\epsilon},\epsilon_G)}, \quad (22)$$

where

$$\chi(R^*;\vec{\epsilon},\epsilon_G) = \frac{(R^*/c + \frac{1}{N}\sum_{j=1}^{N} w_j p_j)^2}{\frac{1}{N^2}\sum_{j=1}^{N} w_j^2 p_j(1-p_j)}. \quad (23)$$

and $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_N)^T$. Since $n \gg 1$, $\delta \ll 1$ and we can apply Laplace's Method. The goal is to reduce the $(N+1)$ dimensional integral in (22) into a single integral in $\mathrm{d}\epsilon_G$ by approximating the $\mathrm{d}\vec{\epsilon}$ integral.

The dominant contribution to (21) comes from points $\vec{\epsilon}^*$ where $\nabla_{\vec{\epsilon}}\chi = 0$, in which case

$$\begin{aligned} f_{R_\pi}(R^*) \approx \frac{1}{\delta(2\pi)^{N/2+1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon_G}{\tilde{\sigma}(\vec{\epsilon}^*,\epsilon_G)} \exp\left[-\frac{\chi(R^*;\vec{\epsilon}^*,\epsilon_G)}{2\delta^2}\right] \exp\left(-\frac{\vec{\epsilon}^*\cdot\vec{\epsilon}^* + \epsilon_G^2}{2}\right) \times \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}\vec{\epsilon} \exp\left[-(\vec{\epsilon}-\vec{\epsilon}^*)^T(\nabla_{\vec{\epsilon}}^2\chi)(\vec{\epsilon}-\vec{\epsilon}^*)/(4\delta^2)\right], \end{aligned} \quad (24)$$

$$\Rightarrow f_{R_\pi}(R^*) \approx \frac{1}{\delta(2\pi)^{N/2+1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon_G}{\tilde{\sigma}(\vec{\epsilon}^*,\epsilon_G)} \exp\left[-\frac{\chi(R^*)}{2\delta^2}\right] \exp\left(-\frac{\vec{\epsilon}^*\cdot\vec{\epsilon}^* + \epsilon_G^2}{2}\right) \times \frac{(2\delta)^N \pi^{N/2}}{(\det\Lambda)^{1/2}},$$

$$= \frac{2^{N/2}\delta^{N-1}}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\frac{\chi(R^*)}{2\delta^2}\right] \exp\left(-\frac{\vec{\epsilon}^*\cdot\vec{\epsilon}^* + \epsilon_G^2}{2}\right) \frac{\mathrm{d}\epsilon_G}{\tilde{\sigma}(\vec{\epsilon}^*,\epsilon_G)\prod_{j=1}^{N}\lambda_j^{1/2}(\vec{\epsilon}^*,\epsilon_G)},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ is the diagonal matrix of eigenvalues of the Hessian matrix $\nabla_{\vec{\epsilon}}^2\chi$

3.5 Determination of $\vec{\epsilon}^*$

Note: The calculation of $\vec{\epsilon}^*$ below (and the subsequent derivation for the portfolio's value) is incorrect. $\vec{\epsilon}^*$ should actually satisfies $R^*/c + (1/N)\sum_{j=1}^{N} w_j p_j = 0$ in (23) (this was assumed to be non-zero and divided out). Nevertheless, the calculations performed at the workshop are reproduced below. An alternative method using the law of large numbers, instead of Laplace's method "Analysis of credit portfolio risk in a localized one-factor model" can be found at `http://udel.edu/~pakwing/publications`.

The stationary point $\vec{\epsilon}^*$ satisfies $\nabla_{\vec{\epsilon}}\chi = 0$ where χ is defined in (23). Therefore we find

$$\left(\frac{2}{N}\sum_{j=1}^{N} w_j^2 p_j(1-p_j)\right)\left(\sum_{j=1}^{N} w_j \nabla_{\vec{\epsilon}} p_j\right) = \left(\frac{R^*}{c} + \frac{1}{N}\sum_{j=1}^{N} w_j p_j\right)\left(\sum_{j=1}^{N} w_j^2(1-2p_j)\nabla_{\vec{\epsilon}} p_j\right), \quad (25)$$

which can be rewritten as

$$\nu_1 \mathbf{A}\vec{w} = \nu_2 \mathbf{A}\vec{u}, \tag{26}$$

where

$$\begin{aligned}
\nu_1 &= \frac{2}{N}\sum_{j=1}^{N} w_j^2 p_j(1-p_j), && (27)\\
\nu_2 &= \frac{R^*}{c} + \frac{1}{N}\sum_{j=1}^{N} w_j p_j, && (28)\\
A_{ki} &= \frac{\partial p_i}{\partial \epsilon_k} = -\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\theta_i - \sqrt{\rho_i}(\beta_G \epsilon_G + \beta_i \epsilon_i)}{\sqrt{1-\rho_i}}\right)\right] \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \beta_i \delta_{ik}, && (29)\\
(\vec{w})_j &= w_j, && (30)\\
(\vec{u})_j &= w_j^2(1-2p_j). && (31)
\end{aligned}$$

Another way of representing $\mathbf{A}$ is $\mathbf{A} = \mathrm{diag}(\vec{\beta})\ \mathrm{diag}(\vec{v})$ where $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)^T$ and

$$(\vec{v})_i = -\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\theta_i - \sqrt{\rho_i}(\beta_G \epsilon_G + \beta_i \epsilon_i)}{\sqrt{1-\rho_i}}\right)\right] \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}. \tag{32}$$

Since $\mathbf{A}$ is non-singular, eq. (26) implies $\nu_1 \vec{w} - \nu_2 \vec{u} = 0$ or

$$\frac{2w_i}{N}\sum_{j=1}^{N} w_j^2 p_j(1-p_j) - w_i^2(1-2p_i)\left(\frac{R^*}{c} + \frac{1}{N}\sum_{j=1}^{N} w_j p_j\right) = 0, \tag{33}$$

for $i = 1, \ldots, N$.

Now specialize to the case $w_j = 1$ for $1 \le j \le N$. Then it is clear from rearranging (33) that $p_j = p$ for $1 \le j \le N$ (all the p_j are identical and independent of j) so that p is given by

$$p = \frac{R^*}{c + 2R^*}. \tag{34}$$

This gives

$$\epsilon_i^* = -\frac{\beta_G \epsilon_G}{\beta_i} + \frac{\theta_i}{\beta_i\sqrt{\rho_i}} - \frac{\sqrt{1-\rho_i}}{\beta_i\sqrt{\rho_i}}\ \Phi^{-1}\left(\frac{R^*}{c+2R^*}\right). \tag{35}$$

3.6 Summary of Solution

$$f_{R_\pi}(R^*) \sim \left(\frac{2}{n}\right)^{\frac{N}{2}} \frac{\sqrt{n}}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\frac{n\chi(R^*; \vec{\epsilon}^*, \epsilon_G)}{2}\right] \frac{\exp\left(-[\epsilon_G^2 + \vec{\epsilon}^* \cdot \vec{\epsilon}^*]/2\right)}{\tilde{\sigma}(\vec{\epsilon}^*, \epsilon_G)} \times \frac{\mathrm{d}\epsilon_G}{\prod_{j=1}^{N} \lambda_j^{1/2}(\vec{\epsilon}^*, \epsilon_G)}, \tag{36}$$

where

$$\epsilon_i^*(\epsilon_G) = -\frac{\beta_G\ \epsilon_G}{\beta_i} + \frac{\theta_i}{\beta_i\sqrt{\rho_i}} - \frac{\sqrt{1-\rho_i}}{\beta_i\sqrt{\rho_i}}\ \Phi^{-1}\left(\frac{R^*}{c+2R^*}\right) \tag{37}$$

$$\tilde{\sigma} = \frac{c}{N}\sqrt{\sum_{j=1}^{N} w_j^2 p_j[\epsilon_j,\epsilon_G]\,(1-p_j[\epsilon_j,\epsilon_G])}, \tag{38}$$

$$\chi(R^*;\vec{\epsilon},\epsilon_G) = \frac{(R^*/c+\frac{1}{N}\sum_{j=1}^{N} w_j p_j)^2}{\frac{1}{N^2}\sum_{j=1}^{N} w_j^2 p_j(1-p_j)}. \tag{39}$$

as the number of companies in each sector $n \to \infty$. The λ_j are the eigenvalues of $\nabla_{\vec{\epsilon}}\chi$.

4 Saddle Point Method

Consider a portfolio with assets distributed in N sectors, with N_i assets in the i-th sector. Let's denote the return of theportfolio as R_π, a random variable. The problem we are interested is to find, for a given cumulative probability α , the the threshold R^* such that

$$P(R_\pi < R^\star) = \alpha. \tag{40}$$

The portfolio return is a weighted sum of the individual returns :

$$R_\pi = \sum_{i=1}^{N}\sum_{j=1}^{N_i} \hat{w}_{ij} R_i(z_{ij})$$

where the $\hat{w}_{ij}$ are normalized weights for each company in the portfolio, $\hat{\omega}_{i,j} \geq 0$ and $\sum_{i,j}\hat{\omega}_{i,j} = 1$. R_i is the potential return of assets in sector i, defined by

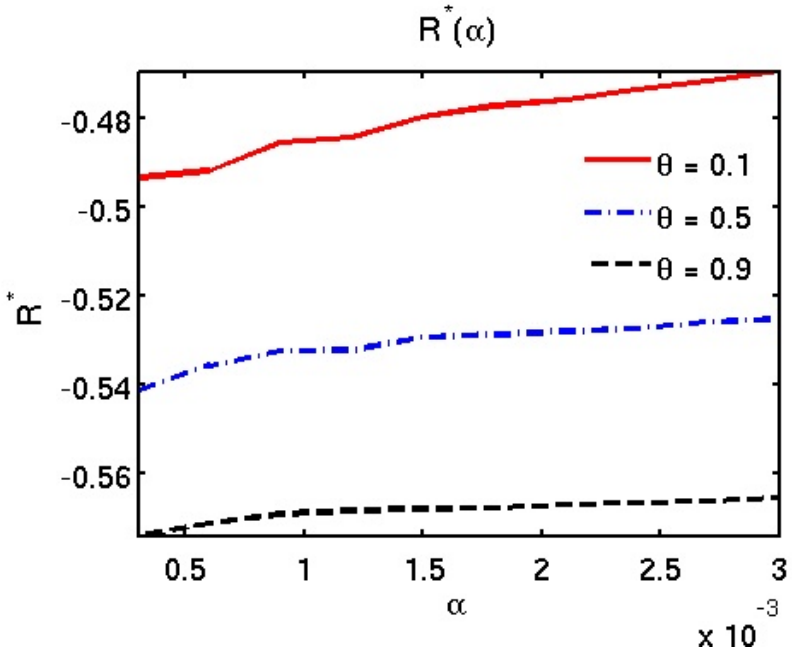
$$R_i(z) = \begin{cases} 0, & \text{if } z > \theta_i \\ -LGD, & z \leq \theta_i \end{cases}$$

θ_i is the default threshold, a known parameter. LGD, loss give default, is a random variable which is assumed to follow a uniform distribution on $[0,1]$ for simplicity. This is to say, if the return z is greater than θ_i, the loss is zero, if it's less than θ_i, the loss is a random variable LGD.

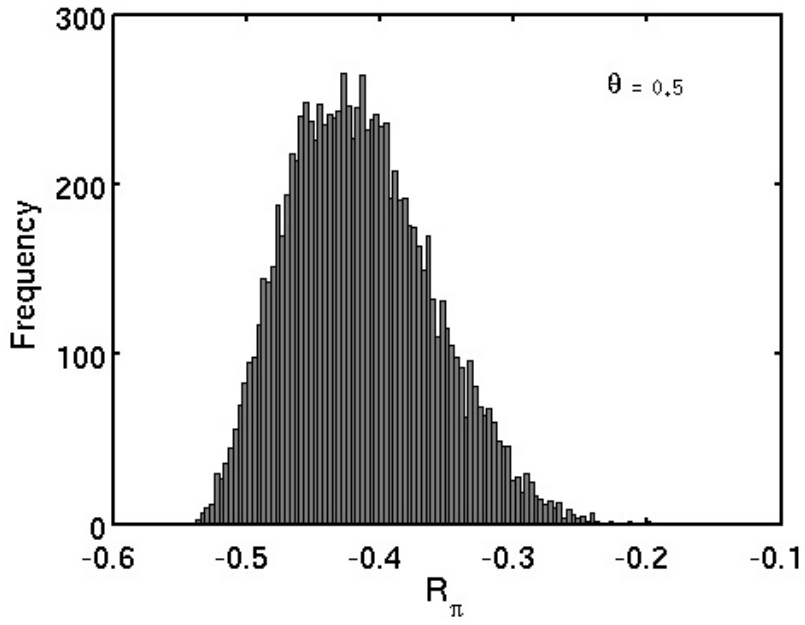
By the localized one-factor model, each z_{ij} is a latent variable associated to the jth company in section i, and is decomposed into global risk factor ϵ^G, the sector risk factor ϵ_i and the idiosyncratic risk factor ϵ_I:

$$z_{ij} = \sqrt{\rho_i}\beta_{ij}^G\epsilon^G + \sqrt{\rho_i}\beta_{ij}\epsilon_i + \sqrt{1-\rho_i}\epsilon_{I,i,j}$$

ϵ^G, ϵ_i, $\epsilon_{I,i,j}$ are all assumed to follow standard normal distribution. $\rho_i, \beta_{ij} \in [0,1]$ are weight parameters, and $(\beta_{ij}^G)^2 + \beta_{ij}^2 = 1$.



(a)



(b)

Figure 2: Results from Monte Carlo simulations. Parameter values were $N = 20$ sectors, $n = 1000$ companies in each sector, $c = 0.6$ and sector independent $\theta_i = \theta$ for $i = 1, \ldots, N$. (a) Threshold loss R^* as a function of α with $\theta = 0.1, 0.5, 0.9$: see eq. (9). (b) Frequency distribution of portfolio value with $\theta = 0.5$.

The problem is equivalent to solving a nonlinear equation $f(R^*) = 0$ for unknown R^*, where $f(R^*) = P(R_\pi < R^\star) - \alpha$ and α is given. We can obviously use , e.g., bisection method to solve it, as long as we know how to evaluate the function f, that is, how to compute $P(R_\pi < R^\star)$ for a given R^*. One way to find $P(R_\pi < R^\star)$ is to use the "Saddlepoint Approximation" method. Saddlepoint approximation is well known to provide good approximations to very small tail probabilities, which makes it a very suitable technique in the context of portfolio credit loss.

Since saddlepoint approximation method utilizes moment generation functions, it can only deal with sum of independent random variables. However, R_π is a sum of DEPENDENT random variables R_i, so we cannot apply saddlepoint approximation method directly. We can get around this problem by consider R_π conditioned on ϵ^G and ϵ_i's instead. This is because assets z_{ij} and z_{kl} are dependent with each other through the global factor ϵ^G, and also ϵ_i if $i = k$, i.e., these assets are both within section i. In other words, $z_{ij}|\epsilon^G, \epsilon_1, ..., \epsilon_N$'s are independent. Therefore, we can apply the saddlepoint approximation to $\tilde{R}_\pi := R_\pi|\epsilon^G, \epsilon_1, ..., \epsilon_N$:

$$\tilde{R}_\pi = \sum_{i=1}^{N}\sum_{j=1}^{N_i} \hat{w}_{ij}\tilde{R}_{ij}, \qquad \text{where } \tilde{R}_{ij} := R_i(z_{ij}|\epsilon^G, \epsilon_1, ..., \epsilon_N) \tag{41}$$

and the tail probability is then computed as

$$P(R_\pi < R^\star) = \int_{\mathbb{R}} P(\tilde{R}_\pi < R^\star) g(\epsilon^G) g(\epsilon_1)...g(\epsilon_N) d\epsilon^G d\epsilon_1 ... d\epsilon_N$$

where g is the probability density function of a standard normal random variable. This integral can be done using normal quadratures without too much difficulty, since the number of sectors N is a lot smaller than the number if assets $\sum_{i=1}^{N} N_i$.

According to the saddle point approximation methods, the conditional tail probability can be approximated by

$$P(\tilde{R}_\pi < R^*) = \begin{cases} 1-a, & \text{if } R^* > E(\tilde{R}_\pi) \\ 1/2, & \text{if } R^* = E(\tilde{R}_\pi) \\ a, & \text{if } R^* < E(\tilde{R}_\pi) \end{cases} \tag{42}$$

where

$$a = \exp(K(\tilde{t}) - \tilde{t}R^* + \frac{1}{2}\tilde{t}^2 K''(\tilde{t}))\Phi\left(-\sqrt{\tilde{t}^2 K''(\tilde{t})}\right),$$

the cumulant generating function of $\tilde{R}_\pi$: $\quad K(t) = \log M(t)$,

the moment generation function of $\tilde{R}_\pi$: $M(t) = E(e^{t\tilde{R}_\pi}) = \prod_{i=1}^{N}\prod_{j=1}^{N_i} M_{\hat{w}_{ij}\tilde{R}_{ij}}(t)$,

$\tilde{t}$ is the saddle point, which is the unique solution of the equation $K^{'}(t) = R^*$,

Φ is the cumulative probability of standard normal distribution ,

The saddlepoint approximation formula given by (3) is proved to have error $O(n^{-1})$ if independent random variables $\hat{\omega}_{ij}\tilde{R}_{ij}$ are identically distributed. There are also higher order saddle point approximation of the tail probability if one look for better accuracy. Note that saddle point approximation method can be readily applied to more general Bernoulli mixture models (possibly multi-factor models).

In the following numerical experiment, the parameters ρ_i, β_{ij}, β^G and θ_i are artificially picked, and for simplicity the normalized weights $\hat{\omega}_{ij}$ are equal, so as θ_I's. We also take the LDG as uniform distribution on [0,1], so that the function K can be computed explicitly. Note that there the parameter $E(\tilde{R}_\pi)$ can be computed without difficulty, since $z_{ij}|\epsilon^G, \epsilon_1, ..., \epsilon_N$'s in (2) are independent normal random variables. The saddle point $\tilde{t}$ is computed numerically by $K^{'}(t) = R^*$. The first plot is a histogram of the probability density function of R_π using Monte Carlo simulation. Recall that the distribution if R_π is the hard to find analytically since it's a sum of dependent random variables. The second plot shows the dependence of tail probability α on loss percentageR^*, for different choice of θ, using saddle point approximation method. For a fixed tail probability α, the larger θ is, more possible it is to default, and thus the smaller R^* should be. Also it's obvious that R^* should be nondecreasing in α. In the last plot ,we use both full Monte Carlo simulation and the saddle point approximation method to plot the dependence of tail probability α on loss percentageR^* . For the conventionally chosen tail probability value $\alpha = 0.001$, the saddle point approximation method gives R^* value 0.24.

References

[1] W. Morokoff, L. Yang, and S. Islam, Approximating Correlation Matrices.

[2] L.Anderson, J.Sidenius, and S. Basu, All your hedges in one basket, Risk, November 67-72, 2003/

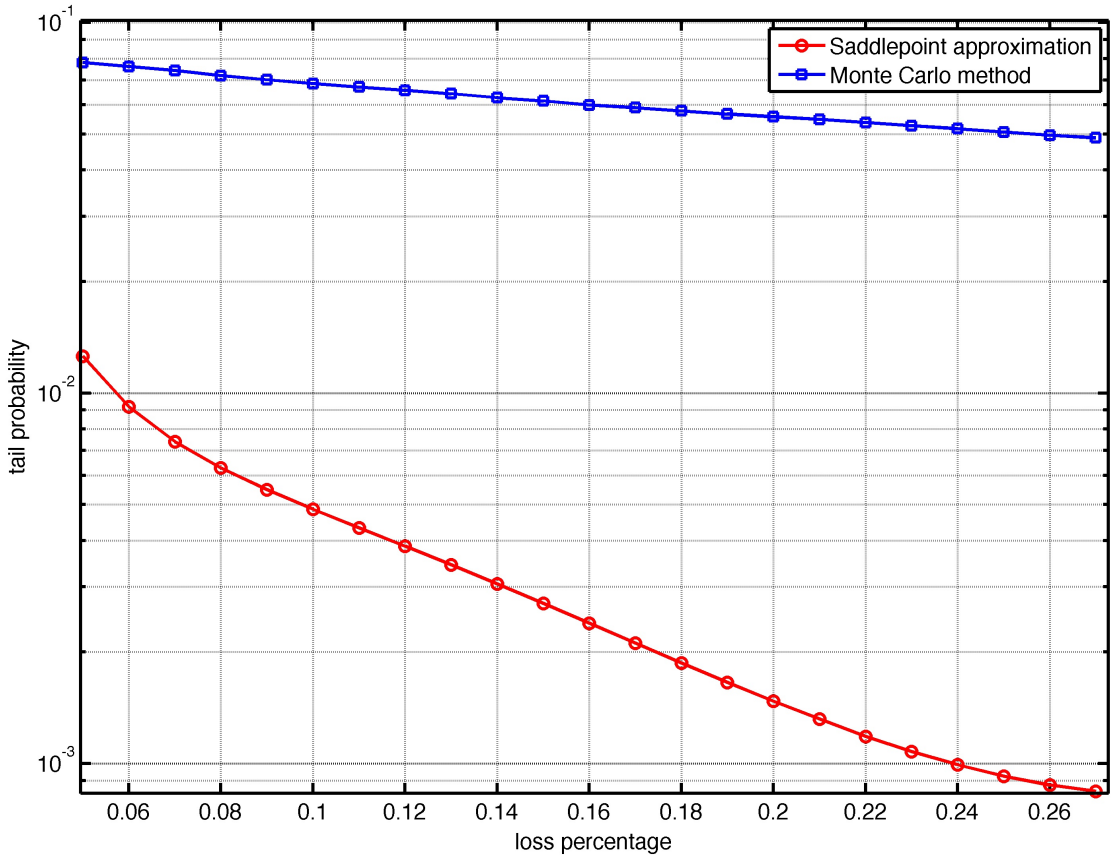


Figure 3: Tail probabilities are calculated using the saddlepoint method.